A SHORTER GEOMETRY

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PREFACE

THIS volume has been prepared to supply a book of reasonable length on School Geometry, and to comply with the recommendations issued in the *Report of the Assistant Masters' Association.*

It has been founded on the authors' *Geometry for Schools*, but the subject-matter has been entirely rearranged by one of the authors, the more difficult parts being omitted, and new matter, together with new sets of examples, introduced.

The *Report on the Teaching of Geometry in Schools* by a small committee of the Mathematical Association was issued when the MSS. of this volume was practically completed. In agreement with one of their contemporaries, the authors are of the opinion that " many teachers are seriously bewildered by this Report " ; amongst other things they do not consider that an axiom of similarity is preferable to one of parallels, the general train of reasoning involved being very difficult, nor would they entirely exclude the method of superposition ; as Prof. H. F. Baker says, " Is not the test of equal length, for schoolboys at least, the possibility of the superposition of the same measuring rod " ?

The first portion of the volume deals with the Fundamental Concepts of Volume, Surface, Dimensions, Lines, Points, with illustrations from the construction of Solids.

This is followed by a section on Horizontal and Vertical Lines, first ideas on Parallels, and Angles.

The measurement of straight lines and angles is then given with simple Constructions *without proof.* The student is thus in the position to draw diagrams to scale and work out simple problems on Heights and Distances.

At this stage, for convenience, various axioms and definitions are collected together followed by a group of important Geometrical Facts on Angles at a point, Parallel Lines and Playfair's Axiom, Congruence of Triangles,—these facts being established by intuition and experiment.

A few easy *logical proofs* are then given on the Angles of a Triangle and Polygon, and also on the Isosceles Triangle. There follow sections on Symmetry, Similarity and the Diagonal Scale, and simple Constructions *with proofs.*

After this point has been reached the propositions are arranged in groups, and, as occasion arises, further Constructions are given.

This final part contains sections on Inequalities, Parallelograms, Loci (examples first being worked out by plotting points satisfying given conditions), Areas of Triangles and Parallelograms, Pythagoras (with examples on solid figures), Circle, Formulae, Extension of Pythagoras, Ratio and Proportion.

The Sets of Examples are in most cases arranged with numerical and construction questions at the beginning. It is most important for the student to realise that something more than a diagram and the answer are necessary in a question on construction ; the method of drawing the figure should be briefly but clearly stated, it is otherwise frequently quite impossible for a teacher or examiner to follow the details of the figure.

In the proofs of the propositions, references are given in words and not by the numbers of previous theorems.

The authors wish to express their thanks to Mr. C. Newton, formerly of Cheltenham College, who has worked through all the examples, and it is hoped that very few errors have escaped detection.

Thanks are also due to the University of Cambridge Local Examinations Syndicate, the Oxford and Cambridge Schools Examination Board, and H.M. Stationery Office, for permission to include questions from various examination papers.

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GEOMETRICAL SOLIDS, SURFACES, ETC.

1. Solids. Objects such as a brick, ruler, golf ball, the glass of a tumbler, etc., are usually called Solids.

In Geometry the word *Solid* is used in a different sense, and means *the portion of space occupied by these bodies.* We speak of the portions of space occupied by such things as the air in a pneumatic tyre, a flame, the water in a tumbler, etc., as Solids.

Solids all have a certain **Volume** (or Size), **Shape** and **Position.**

2. The Volume (or Size) of a Solid is of great practical importance, and must be carefully distinguished from its shape.

The volume of a body may be measured by the rise of water in a graduated flask, as in Fig. 1; if the water is at A to begin with, it will rise to some point such as B when a solid is immersed, and from the difference of these two graduations the volume may be read off.

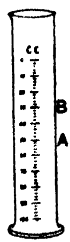


FIG. 1.

Students would do well to test the volumes of several solids of various shapes, such as pieces of granulated zinc, marbles, short pieces of copper wire, etc.; in this way they will learn **the difference between Volume and Shape.**

3. No solids can be accurately drawn on a flat sheet of paper. If a thin piece of string, a foot or so long, is stretched due East and West, and another similar piece stretched a few inches above it due North and South, a fairly good representation of either piece of string can be drawn on a

flat sheet of paper; but they cannot be drawn both together. If the string is very thin, either piece approaches towards something which is no longer a solid and can very nearly be drawn, but the two strings together are such that nobody can draw them with any approach or pretence to accuracy; the artist can give an optical representation of them, using the artifice known as perspective. The two together are said to constitute a figure in **three dimensions**, even though each string represents only one dimension.

4. The question will be asked what is a dimension? Without attempting a full answer, the following may be useful.

If it is wished to mark some point on an edge of a table, it is necessary to mention only **one measurement**, viz. the distance from one end of that edge; **one measurement only** is needed because the edge is of **one dimension.**

If, however, it is wished to mark some point on the surface of the table, **two** measurements must be given, *e.g.* the shortest distance of the mark from **two** adjacent edges; **two** measurements are necessary because the surface of the table is of two dimensions.

If, further, it is wished to suspend some small object at a point in the space above the table, then in addition to two such measurements as the last, a third would be required, viz. the height above the table; **three measurements** are necessary because the space above the table is of three dimensions.

5. Surfaces. If we consider the outside of a closed wooden box, there are certain regions where the wood leaves off and the air begins; these regions are called **Surfaces**; they have no thickness or material existence; the surface is not wood, for the thinnest shaving of wood has some thickness and is a solid; neither is the surface air.

In the same way, considering the *page* of a book, there is a region where the paper leaves off and the air begins; the region is a surface and is neither paper nor air; and again we must carefully distinguish between the geometrical *solid*, though its thickness is very small, and the *surface*.

6. Surfaces like these are called **Plane Surfaces**, but there are others besides; a ball, the curved part of a pencil and the worm of a cork-screw are all bounded by surfaces, but they are different from the others and are called **Curved Surfaces.** It should be noticed that every solid is bounded by one or more surfaces.

Plane and curved surfaces can be distinguished by applying to the surface the straight edge of a ruler; if it lies entirely in contact with the surface in **every position**, the surface is plane.

7. The *correctness of the straight edge* of a ruler can be tested thus:

Take any two points on paper and rule a line joining them, placing the ruler on one side of the points; then turn the ruler round and again rule the line, the ruler being on the other side of the points, for correctness, the two ruled lines must be absolutely the same.

8. **Lines.** A side and the top of a box meet in what is commonly called an edge; in Geometry we call this a line.

The curved surface of a round ruler meets the flat end in a line; but this is a different kind of line from the first; it is called a *curved line*, whereas the first is called a *straight line.*

Wherever two surfaces meet there is a line, sometimes straight, sometimes curved.

It should be observed that we cannot accurately draw a line either with a nib or pencil, since, however finely they are used, that which is drawn will always have a certain amount of breadth.

A line has no breadth and no thickness.

9. **Points.** Consider the case of a box again, and look at one of the top corners; it will be seen that two top edges meet there; the place of meeting of two lines is called a **Point.** It will also be noticed that a third edge passes through the point as well; any number of edges or lines may meet at a point, but two lines are sufficient to indicate a point.

We might also consider the above point as the place where the third edge mentioned meets the top surface, for the meeting place of a surface and a line is a Point.

It should be noticed that we cannot draw a point; however small a dot is made with ink or pencil, it has a certain size and is not accurately a point.

A point has no length and no breadth and no thickness.

In cases where accurate measurements have to be made,

Fig. 2.

it is best to represent points by drawing two short thin intersecting lines and to avoid dots.

10. A familiar example of a curved line is the circumference of a **circle**, in which every point is at the same distance from a certain point within the figure; this point is called its centre.

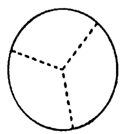


Fig. 3.

EXERCISE 1.

1. Name, from objects in the room, some examples of surfaces, lines and points

2. Name some solids which have all their surfaces plane.

3. Name any familiar solid in which the surface is entirely curved.

4. If you were asked how much of the wall of a room was to be papered, would you give the answer as a volume, area or length?

5. Which are the plane and curved surfaces in the ordinary form of a class-room?

6. Which are the plane and curved surfaces in

(*a*) a tumbler, (*b*) a cigarette, (*c*) a new pair of shoes?

7. In the following cases, would you give the answer as a volume, area or length?

(*a*) The amount of ink in the ink-well.
(*b*) The quantity of paper for a parcel.
(*c*) The amount of acid for a chemical experiment.
(*d*) The amount of darning-cotton for mending a sock.
(*e*) The amount of zinc for covering a roof.
(*f*) The quantity of gas burnt in a flame in 1 hour.

8. Is a railway-line truly a line, and are railway-points really points?

11. The Figs. 4-10 represent Solids of various shapes. The student should look at these objects themselves; he

will gain in this way a better idea of them than from figures.

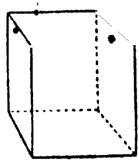


FIG. 4.

Fig. 4 represents a **Rectangular Block** or **Cuboid**; it has

6 plane surfaces, 12 straight edges or lines, 8 angular points.

Fig. 5 represents a **Sphere**; it has

1 curved surface, no edges and no points.

Fig. 6 represents a **Pyramid**; it has

5 plane surfaces, 8 straight edges or lines, 5 angular points.

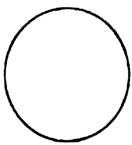


FIG. 5

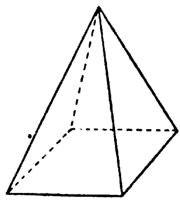


FIG. 6.

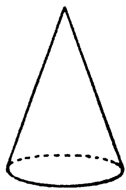


FIG. 7.

Fig. 7 represents a **Cone**; it has

1 curved surface, 1 plane surface, 1 curved edge or line, and 1 angular point.

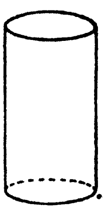


FIG. 8.

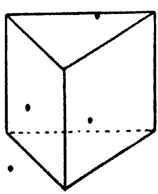


FIG. 9.

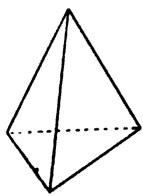


FIG. 10.

Fig. 8 represents a **Cylinder**; it has

1 curved surface, 2 plane surfaces, 2 curved edges or lines.

Fig. 9 represents a **Triangular Prism**; it has

5 plane surfaces, 9 straight edges or lines, and 6 angular points.

Fig. 10 represents a **Tetrahedron** ; it has

4 plane surfaces, 6 straight edges or lines, and 4 angular points.

12. Direction. Tie a small weight to the end of a thin string and hold up the weight by the string ; the string is then said to be **Vertical**. It represents a line in a **Vertical Direction.**

How many vertical lines are there in a room ?

Hang the weight in many different places. Do you think that a **Vertical line** is always in the **same direction** ?

The edges the walls form with each other are (or ought to be) vertical ; so are some of the edges of the doors and of the windows.

Mention as many vertical lines as you can.

Does the weight always hang in precisely the same way when hung from the same point ; do you think more than one vertical line can be drawn through a point ?

If you hold the string against a wall and the weight hangs as in Fig. 11, would the wall be correctly built ?

Fig. 11.

Could you draw any vertical lines down such a wall ?

The above illustrates the principle of the *builder's plumb-line.*

Do you think you could draw any vertical lines on a sloping black-board or on the floor ?

13. The edges where the walls meet the floor or the ceiling are **Horizontal**.

Hold out your arm horizontally ; does it keep horizontal if you slowly turn round ?

The plane which contains all the horizontal lines which can be drawn through a particular point is a *Horizontal Plane.*

Such a plane may easily be tested by means of a *Spirit Level,* which may be either a *long* one or a *round* one such as photographers employ (Fig. 12).

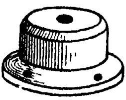


Fig. 12.

In either case the bubble is in the middle when the spirit level is placed on a horizontal plane.

In using the *long spirit level* care must be taken to use it in *two directions.*

14. The direction of a plumb-line when produced passes through the centre of the earth, and thus a plumb-line in London would not be in the same direction as one in Edinburgh; but if there are several plumb-lines close together, then the **vertical lines** which they indicate may be considered to be in the **same direction.**

15. Horizontal lines, however, are not necessarily in the same direction. If everybody in a class-room points his arm straight upwards, all the arms are vertical and in the same direction; but if everybody points his arm horizontally at a tall person standing in the middle of a room, though all are pointing horizontally, they are certainly not pointing in the same direction.

Now settle which is the southerly aspect of the class room. What is an easy method of determining this direction?

Having settled on this, you know at once the northerly and other aspects.

Let everybody point his arm in the northerly direction (of course the arms should be horizontal), then all the arms will be seen to be in the same direction.

Any two such lines are called in Geometry **Parallel lines**, and possess the property of *never meeting, however far they are produced.*

16. The printed lines on a sheet of an exercise book, and the two sets of lines on a sheet of squared paper, are examples of parallel lines in the same plane, a, b, c, d, e forming one set of parallel lines and p, q, r, s, t another set.

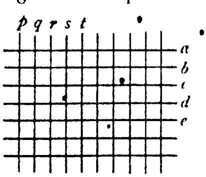


FIG. 13.

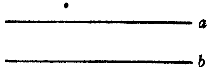


FIG. 14.

Though we have not yet considered the subject of the measurement of lines, it will probably easily be recognised

that any two parallel lines, such as *a* and *b* (Fig. 14), are everywhere the same distance apart, as, for instance, in the case of the two rails of a piece of straight railroad.

17. Any two vertical lines are parallel. The crevices between the planks in the floor are in the same direction or parallel.

The wickets on a cricket pitch are all vertical and parallel.

The markings on a tennis lawn are all horizontal, but they are not all in the same direction nor parallel. If the top string of the net is E. and W., then the side lines (single and double courts) are in the same direction N. and S. and are parallel, and the other lines are in the direction E. and W. and are parallel.

EXERCISE 2.

1. Draw a rough sketch of Rugby football goal posts and mark which of the posts or bars are horizontal, and which vertical.

2. Name some common instances of parallel lines.

3. Can you draw a vertical line on a sloping railway embankment? If so, how many?

4. Can you draw a horizontal line on a sloping railway embankment? If so, how many?

5. Through any particular point on a sloping roof, how many vertical lines and how many horizontal lines can be drawn?

6. If a page of a book is held with a line of printing horizontal, will the long edge of the page be vertical?

7. If a flat ruler is held with its length vertical, will its breadth be horizontal?

8. Draw a rough sketch of a flight of stone steps and mark the horizontal and vertical lines.

18. Angle. Two non-parallel lines in the same plane, that is, two straight lines which intersect, are said to form an angle with each other.

FIG. 15.

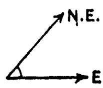


FIG. 16.

Point one arm N. and the other E. Your arms are inclined at an angle; this angle is called a **Right angle** (Fig. 15).

Point the first arm N.E. instead of N., and now your arms are inclined at a different angle (Fig. 16).

How does this second angle compare with the first?

Again point one arm N. and the other S.E., and you have to stretch your arms much wider apart than in the first case, and they form a greater angle.

Again point your arms N. and E., and think of the angle between them. Then point the second arm vertically upwards instead of E.

How does this second angle compare with the first?

Stand facing N., then *right turn*; you will now face E. You have turned through a **Right angle.**

It is most important to be familiar with a right angle; it is easy to recognise and may be divided into 90 equal parts called **Degrees**; *i.e.* the turn from N. to E. may be made by 90 equal smaller turns, each equal to one degree.

Again point one arm N. and the other N.E.; the angle between your arms is half a right angle (45 degrees).

Again stand facing N. and *half-right turn*. What direction do you now face? Through what angle have you turned?

Face N. and *about turn*. You will be facing S., and will have turned through 2 right angles (180 degrees).

19. The directions of the hands of a clock form an angle; at 3 and 9 o'clock the angle between them is a right angle; at 12 o'clock the angle is nothing; at 6 o'clock the angle is the same angle as that which you turn through when you *about turn*, that is two right angles; at 16 mins. past 6 o'clock the hands are approximately at right angles. You will notice all these angles are the same whether you are looking at a watch with very small hands or a clock with very large ones.

20. The two legs of an opened compass form an angle. Hold the compass leg OA still and rotate the other one OB. The angle between these legs does not depend in any way on the size of the compasses; you may have a very small pair or you may have large ones as used for a blackboard. But

FIG. 17.

the angle does depend on the **amount of rotation** that you give to OB.

The angle above is spoken of as $\hat{AOB}$ or $\hat{BOA}$; or sometimes as $\hat{a}$ or $\hat{O}$ when there is no chance of confusion.

21. If two angles are very different in size, you can tell there is a difference by observation; for instance, $\hat{HOK}$ is clearly a larger angle than $\hat{RQS}$.

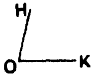


Fig. 18

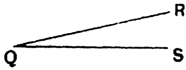


Fig 19.

If there is some doubt as to whether two angles AOB, CQD are equal (Figs. 20, 21), cut out two pieces of paper so that the edges fit on the *arms* of the two angles; then place one piece of paper on the top of the other and determine whether the edges coincide as in Fig. 22; if so, then the angles are equal.

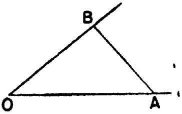


Fig. 20.

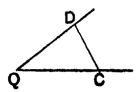


Fig. 21.

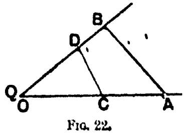


Fig. 22.

It should be noticed that the actual pieces of paper are by no means equal.

22. To construct a **Right angle.**

Take a piece of paper AKBQ, having AB a straight line (Fig. 23). Fold the paper so that the line KB exactly falls

along the direction of the line KA (Fig. 24). A crease is made, shown in Fig. 23 by the dotted line KM.

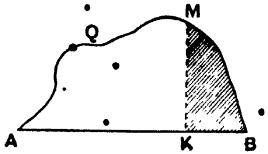


FIG. 23.

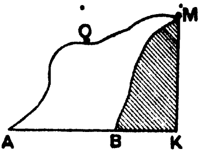


FIG. 24.

The angles $M\hat{K}B$ and $M\hat{K}A$, which are equal, are both right angles.

If you wish to test an angle such as that at the corner of a sheet of paper, a piece of folded paper similar to that shown in Fig. 24 will be found convenient.

EXERCISE 3.

Give approximately the sizes of the angles between the hands of a clock *in terms of 1 right angle*, drawing rough diagrams :

1. At 5 minutes past 4 o'clock.

2. At 7 minutes past 3 o'clock

3. At 2 minutes to 4 o'clock.

4. At 6 o'clock.

5. At 10 minutes to 6 o'clock

6. At 16 minutes past 11 o'clock

7. A boat is sailing due N. and changes its course to N.E. ; through what angle does it turn ?

8. A boat changes its course from N E through E. to S. ; through what angle has it turned ?

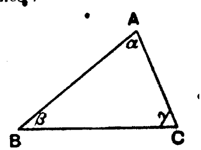


FIG. 25.

9. In Fig. 25, name each of the angles α, β, γ by three letters.

10. In Fig. 25, name the pair of lines containing each of the angles at A, B, C.

11. Name by three letters each of the angles 1, 2, ..., 13, in Fig. 26 in as many ways as possible.

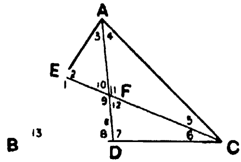


FIG. 26.

12. Of the three angles *a*, *b*, *c*, guess which is the largest and which the smallest; then trace on a piece of thin paper, cut them out and test your result by fitting them on one another.

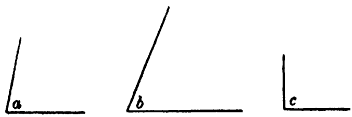


FIG. 27.

MEASUREMENT OF STRAIGHT LINES AND ANGLES.

23. When it is required to measure the distance between two points by means of dividers, the ruler graduated in centimetres and millimetres or in inches and tenths of an inch is used as follows:

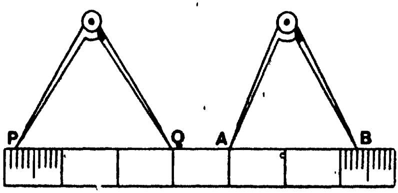


FIG. 28.

Place a point of the compass dividers on each of the given points, then apply the dividers to the ruler as in Fig. 28.

The length AB would be recorded as 2·3 cms. or 2 cms. 3 mms. The length PQ is greater than 2·8 and less than 2·9; the student must try to imagine that the small space at P

is divided into 10 subdivisions and estimate at which subdivision he thinks P would be, in this case perhaps at the third; then the measurement would be 2·83.

24. If the ruler is used direct without the aid of dividers, it should be tilted so that the actual graduation marks come into *contact with the surface* on which lies the distance AB.

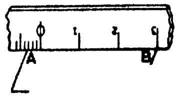


FIG. 20.

Note that an exact graduation is placed over one extremity B, and that the smaller graduations come over the other extremity A.

It will be seen that AB = 3·34 cms. (approx.).

Why is it necessary to tilt the ruler in order to obtain a correct answer?

25. If a straight line is divided into two equal parts, it is said to be **bisected**.

(i) If a line AB is drawn on paper, fold it so that B comes on A, then, on unfolding, if the line AB meets the crease XY at C, C will be the middle point of the line.

Notice that CB was exactly on the top of CA and that the angles XCB, XCA are right angles.

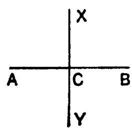


FIG. 30.

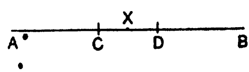


FIG. 31.

(ii) Open out the compasses to any distance AC which looks less than half the line AB, and then mark off BD = AC. If we now estimate the middle point X of the small distance CD, we have the middle point of the line AB.

(iii) An accurate method of bisecting a line AB is as follows:

With centres A and B and radius AB, circles are drawn meeting in C and D.

If CD cuts AB at E, then E is the middle point of AB.

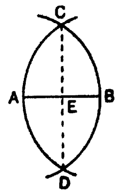


FIG. 32.

EXERCISE 4.

1. Measure AB, BC, CD (Fig. 33) in centimetres and millimetres, add the results together, and then measure AD.

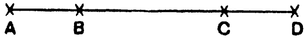


FIG. 33.

2. Measure AD, AC, AB (Fig. 34) in centimetres and millimetres;

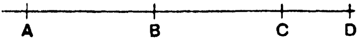


FIG. 34.

subtract the values of AC and AB from that of AD, and compare with the lengths of CD and BD found by measurement.

3. Repeat Qn. 1, giving the measurements to the nearest tenth of an inch.

4. Repeat Qn. 2, giving the measurements to the nearest tenth of an inch.

5. Draw a line AB 5·8 cms long, and guess its middle point C; now cut off from AB a length AD = 2·9 cms, and see how near D is to C.

6. Measure in (i) centimetres and millimetres, (ii) inches and tenths, the total first line of printing of this question.

7. Measure in (i) centimetres and millimetres, (ii) inches and tenths, the length of printing in the second line of Qn. 10.

8. In Fig. 35 measure the lengths of BD, DC; CE, EA; AF, FB; also GD, GA; GE, GB; GF, GC. Give the answers in cms. and mms.

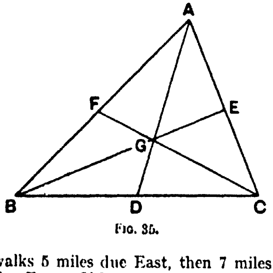


FIG. 35.

9. A man walks 5 miles due East, then 7 miles due West, and finally 9 miles due East. If 1 cm. represents 1 mile, draw a diagram to represent his various positions A, B, C, D. Measure AD in cms.; what distance does this represent?

10. If a man walks 4 miles due North, 5 miles due South, and then 3 miles due North, represent his positions A, B, C, D by drawing a straight line on a *scale* of 1 inch to 1 mile. Measure AD in inches.

11. Draw a triangle with its sides 5 cms., 4 cms., 2 cms. respectively.

N.B.—A *triangle* (△) is a plane figure bounded by three straight lines.

[Draw AB = 5 cms.; with centre A and radius 4 cms. draw an *arc*, *i.e.* a portion of the boundary or circumference of a circle; with centre B and radius 2 cms. draw another arc; let these two arcs cut at the point C, then ABC is the triangle required.]

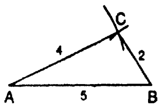


FIG. 36.

12. Draw a triangle ABC such that AB = 3·2 ins., BC = 2·7 ins., CA = 1·8 ins. Take D in AB such that AD = 1·4 ins. Measure CD.

13. Draw a triangle ABC in which AB = 6·4 cms., BC = 5·2 cms., CA = 4·7 cms. In BA, BC take points X and Y such that BX = 4·1 cms., BY = 3·2 cms. Measure XY (Fig. 37).

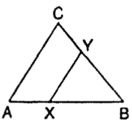


FIG. 37.

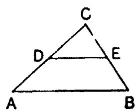


FIG. 38.

14. Draw a triangle ABC, making AB = 6·3 cms., BC = 4·3 cms., CA = 5·7 cms. By intersecting circles, bisect BC, CA, at E, D respectively. Join DE and measure it. See if DE = ½AB (Fig. 38).

15. Draw a triangle ABC in which AB = 5·4 cms., BC = 6·1 cms., CA = 4·6 cms. Bisect BC, CA, AB at X, Y, Z respectively. Join and measure AX, BY, CZ, and show that these three lines meet in a point.

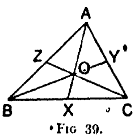


FIG. 39.

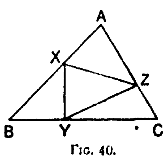


FIG. 40.

16. Draw a triangle ABC, making AB = 3·2 ins., BC = 3·9 ins., and CA = 2·3 ins. In AB, BC, CA take X, Y, Z, so that AX = 1·1 ins., BY = 1·4 ins., CZ = 1 inch. Measure XY, YZ, ZX (Fig. 40).

26. Measurement of Angles. The student has already gained some idea of an angle and of a right angle. The

angle ABC (Fig. 41) is a right angle; as we have seen in Art. 18, it is usual to divide right angles into 90 equal parts, each one of which is called a **Degree**.

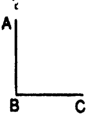


Fig. 41.

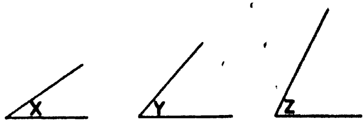


Fig. 42.

The angles X, Y, Z are respectively 30° ; 45° ; 60°.

The student should look well at these and try to recognise them when he sees them again.

Before attempting to measure any angle the student should always **form an approximate idea** of the size of the angle. If he gets into this habit, he will be saved many errors in reading the protractor (Fig. 45).

27. A right angle can be conveniently constructed by means of a **set square**, which may contain angles of 90°, 60°, 30° (Fig. 43).

The set-square can also be used for drawing a line perpendicular to a given line through a given point in it.

(In Fig. 41, BA, which makes a right angle with BC, is said to be perpendicular to BC.)

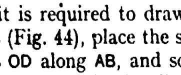


Fig. 43.

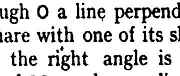


Fig. 44.

If it is required to draw through O a line perpendicular to AB (Fig. 44), place the set-square with one of its shorter edges OD along AB, and so that the right angle is at O; then pencilling in the direction of OC, we have a line perpendicular to AB.

It should be noted that *we can draw only one perpendicular to* AB *through the point* O.

28. The semi-circular **Protractor** is as illustrated in Fig. 45.

To measure an angle KOB, the line P′OP on the protractor is placed along OB with the middle point, which is marked, at O.

The graduation which lies on OK is then read off.

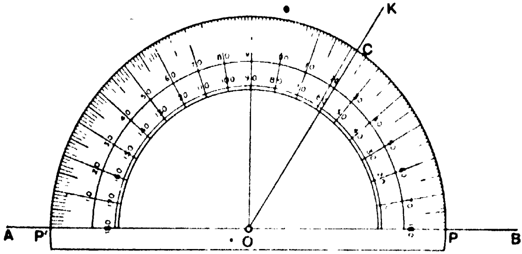


Fig. 45

In this case $\hat{KOB} = 58°$.

A moment's thought will prevent any student from saying that $\hat{KOB} = 122°$; $\hat{AOK}$ is clearly greater than a right angle and $\hat{KOB}$ less than a right angle;

$$\hat{AOK} = 122°.$$

29. To construct an angle of a given size at a given point O in a line OB, the protractor is placed with its centre at the given point O and the line OP along the line OB. If the angle required is 58°, a fine pencil mark is then made at C at the 58° graduation, the protractor removed and OC joined.

EXERCISE 5

1. Measure $\hat{AOB}$, $\hat{BOC}$ and $\hat{AOC}$, and check your work by comparing the third angle with the sum of the first two.

N.B.—The figure may be pricked through on to another sheet of paper.

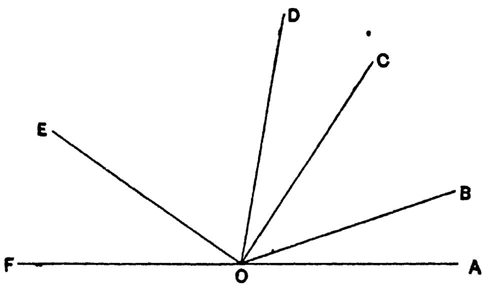


FIG. 46.

2. Measure $\hat{BOC}$, $\hat{COD}$, $\hat{BOD}$ and check

3. Measure $\hat{COD}$, $\hat{DOE}$, $\hat{COE}$ and check

4. Measure $\hat{DOE}$, $\hat{EOF}$, $\hat{DOF}$ and check.

5. Measure $\hat{AOD}$, $\hat{DOF}$ and add your results.

6. Measure $\hat{AOC}$, $\hat{COF}$ and add your results.

7. Measure $\hat{AOE}$, $\hat{EOF}$ and add your results.

8. Measure $\hat{AOB}$, $\hat{BOF}$ and add your results.

Carefully notice the addition in the last 4 questions.

9. Draw an angle of 54°, mark off a point in each arm 3 cms. from the angular point and measure the distance between the points. Join the points and measure the angles formed, and add up all three angles of the triangle.

10. Draw an angle of 16°, mark off a point on one arm 6 cms. and a point on the other arm 4 cms. from the angular point, and measure the distance between the points. Join the points and measure the angles formed, and add up all three angles of the triangle.

11. Repeat Qn. 10, using the numbers 22°, 5 cms., 7 cms.

12. Repeat Qn. 10, using the numbers 38°, 6 cms., 5 cms.

13. Draw two angles AOB, BOX each equal to 33°, produce AO to Y, making OY = 3 cms.; cut off OX = 3 cms., and join XY. Measure the angles OXY, OYX (Fig. 47).

14. What is the number of degrees described by the minute hand of a clock when it moves from figure (i) 12 to 3, (ii) 12 to 4, (iii) 3 to 8, (iv) 2 to 9?

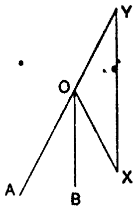


FIG. 47.

15. Draw a straight line AOB, and then make $\hat{BOC} = 54°$. Produce CO to D and measure the angles AOC, AOD, BOD.

16. Draw a triangle with sides 6·2 cms., 5·3 cms., 4·7 cms. Measure the three angles and find their sum.

17. Construct a triangle with sides 3·2 ins., 2·7 ins., 2·2 ins., and measure the three angles. What is the sum of the angles?

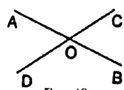


FIG. 48.

18. A man sets out from A and walks 2 miles to B in an Easterly direction, then continues 2 miles to C in a North-Easterly direction, then 3 miles to D in a Northerly direction. Draw a diagram to scale, and find by measurement the distance from A to D and the angle BAD.

19. Draw two lines OA and OB, each 3 inches long, including an angle of 70°. From A, and on the same side of OA as B, draw a straight line AP such that the angle OAP is 100°. Find two points in AP, each 2¾ inches from B, and measure the distance between them.

[Open out the compasses to 2¾ ins. and draw part of a circle with centre B.]

20. Draw a triangle ABC in which AB = 2·3 ins., BC = 2·15 ins., CA = 2·7 ins.

Find the point Q inside the triangle such that each of the angles ABQ, BCQ is 29°. Join AQ, and measure AQ and the angle QAB.

21. Draw the triangle ABC, having BC = 3 ins., and the angles B and C each 65°. In BC take BP = 1 in.; in CA take CQ = 1 in.; and in AB take BR = 2 ins. Show that the triangle PQR has two sides equal.

22. Draw a triangle ABC with AB = 6·5 cms., BC = 5·4 cms., CA = 4·3 cms. Bisect the sides AB, BC, CA at X, Y, Z; through X, Y, Z, draw perps. to AB, BC, CA respectively; see if these perps. meet at a point O; then measure OA, OB, OC, OX, OY, OZ.

23. In the piece of machinery shown, B moves backwards and forwards along the line BO, while A describes the circumference of a circle with fixed centre O. Draw the circle to scale with OA = 2·5 ft.

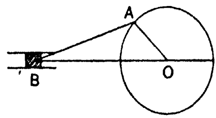


Fig. 49

If AB = 6 ft. and the values of the angle BOA in succession are 0°, 30°, 60°, 90°, 120°, 150°, 180°, measure in each case the value of OB.

30. To bisect a given angle.

If XOY is the angle, draw a circle with centre O and any convenient radius to cut OX, OY at A and B respectively.

With centres A and B and equal radii, draw arcs cutting at P. Then OP bisects the angle XOY (Fig. 50).

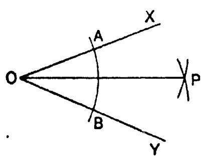


Fig. 50.

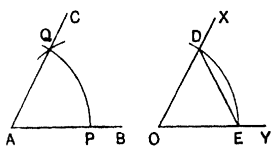


Fig. 51.

31. To draw an angle equal to a given angle.

Let XOY be a given angle, and suppose it is required to draw at A a line AC making with a given line AB an angle BAC = angle XOY.

With centre O and any radius draw an arc of a circle cutting OX, OY in D, E respectively.

With centre A and the same radius draw an arc cutting AB at P.

Open out the compasses to a distance ED, and then with centre P and radius equal to ED draw an arc meeting the previous arc at Q. Join AQ.

The angle QAP = angle XOY.

32. Through a given point to draw a straight line parallel to a given straight line.

Let P (Fig. 52) be the given point and AB the given straight line.

Place a set square with the long edge along AB and a ruler close up against the shortest edge.

Hold the ruler firm and slide the set-square along it until the long edge passes through P, then draw A′B′ along this long edge.

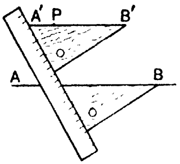


FIG. 52

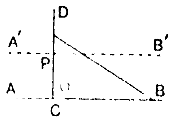


FIG. 53

It will be noticed that, since the short edge has moved along the same straight line all the time, the long edge has continuously been pointing in the same direction and A′B′ is parallel to AB.

Note that through the given point P we can only draw one straight line parallel to the given straight line AB.

If the line A′B′ is to be a given distance from AB, say 1 inch, place the set-square with one of the edges containing the right angle along AB (Fig 53), and draw CD at right angles to AB along the other edge. On the line CD mark a point P so that CP = 1 inch, then through P draw A′B′ parallel to AB by the method given above.

33. To draw a line perpendicular to a given line from a given point without it.

Let AB be the given line and O the given point.

Place the set-square so that one of the shorter edges passes through O and the other short edge lies along the line AB.

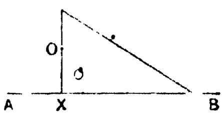


FIG. 54.

Then pencil in the perpendicular OX.

Note that through O only one perpendicular can be drawn to AB.

EXERCISE 6.

1. Draw an angle XAY = 46°. Measure AB, AC each equal to 4 cms.; join BC and, with compasses and ruler, bisect BC at P. Join AP and then measure the angles BAP, CAP with a protractor (Fig. 55).

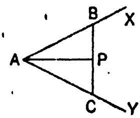


Fig. 55.

2. On a straight line AB take any point O, and from O draw a straight line OC in any direction.

Bisect the angles AOC, BOC by the straight lines OP, OQ, and with a protractor measure the angle POQ.

3. With the protractor make an angle of 39°. Draw a line AB 6·9 cms. in length Without using the protractor again make an angle BAC = 39°. Cut off AC = 5·7 cms, and join CB. Bisect the angle ACB, and measure in centimetres the length of the bisector from C to the point where it cuts AB

4. Construct a triangle ABC such that the angle ABC = 60°, ACB = 75°, and the base BC is 4 centimetres in length. Bisect the angle A by a straight line AX meeting BC in X. Measure the length of BX and CX.

5. Describe a triangle ABC such that AB = 3·3 inches, BC = 2·4 inches, and CA = 1·5 inches Bisect the angles of this triangle by straight lines drawn to meet the opposite sides.

6. Construct an angle of 60°, BAC, making the arm AB = 2½ inches and the arm AC = 2 inches. Join BC. Bisect the angle ABC by a line meeting AC at D Measure AD and DC.

7. Draw a triangle ABC having its base BC 2 inches long and AB, AC each 3 inches long

Bisect the angle ABC by a straight line meeting AC at D; bisect BD at E. Join AE, and measure its length.

8. Draw a line AB 4·7 inches long. Construct at B an angle ABC equal to 60°, and from a point P on BC 2·2 inches distant from B draw PQ perpendicular to AB. Measure PQ. Join PA and measure the angle APQ with a protractor.

9. Bisect at C a straight line AB, 2·75 inches long and draw CD at right angles to AB, and 1·5 inches long; through D draw EDF parallel to AB.

10. Draw an angle BAC of 110° and construct its bisector. Make AB 2·3 inches long and at B construct a perpendicular to AB meeting the bisector of the angle in O. From O construct a perpendicular to AC meeting it in C. Measure OB and OC.

11. On a base 2½ inches in length describe an equilateral triangle ABC. Bisect AB in D and BC in E. Through D and E draw parallels, to BC and BA respectively to meet the third side AC.

12. Draw two straight lines parallel to a given straight line, and each distant one inch from it; draw a straight line PQR cutting them in the points P, Q, R. Show that PQ = QR.

13. Construct a triangle ABC in which A = 56°, AB = 8·7 cms., AC = 5·4 cms. Draw BE bisecting the angle ABC and CF bisecting the angle ACB. Let BE meet CF in O, and AC in E. Measure the angles BEC, BOC.

14. Draw two parallel lines, AB, CD, such that the perpendicular distance between them is 5 cms. Take any point P between them, and through P draw a line LM, cutting AB at L and CD at M, so that LM = 7 cms.

15. In a triangle ABC, the lengths of BC, CA, AB in order are 4·8, 6·3 and 7·1 cms. AD is drawn perpendicular to BC. Measure the lengths of BD, DC.

16. Draw a triangle ABC having AB = 6·8 cms., BC = 5·3 cms., angle ABC = 132°. From C draw CD perp. to AB produced, and measure CD.

17. Draw a line KT, 5·1 inches in length; at K make an angle HKT equal to 39°, and make KH equal to 3·6 inches; draw HS perp. to KT. Measure HS. Join TH and measure the angle THS.

18. To find the breadth of a river with a fairly straight bank, a distance AB is measured along the bank and found to be 150 ft. O being an object on the farther bank, the angles OAB, OBA are measured and found to be 62°, 45° respectively. By drawing the triangle OAB to scale and measuring the length of the perp. from O to AB, find the breadth of the river.

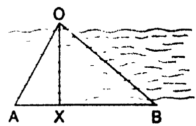


FIG. 56.

19. To find the height of an accessible object AB, suppose CB is the distance of the eye from the ground (5 ft.), and walk a distance CD away from the object. By observation, let $A\hat{D}C = 32°$, and by pacing CD = 240 ft.; $A\hat{C}D = 90°$. Draw the triangle to scale and thus find AC, and then AB (Fig. 57).

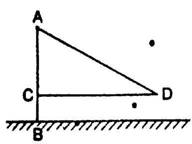


FIG. 57.

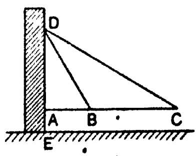


FIG. 58.

20. To find the height of an inaccessible vertical object DE, observe the angle DBA at some point B, walk directly away from the object to C and observe angle DCB. If $D\hat{B}A = 65°$, $D\hat{C}B = 28°$, BC = 210 ft., AE = 5 ft., draw the △DBC to scale, then draw DA perp. to CB produced, and thus find DE (Fig. 58).

34. The Compass. The direction of one point when seen from another can often be given by the reading on a magnetic compass. It has to be remembered, however, that the N. point is not fixed in direction, but varies from year to year; in 1923 the magnetic N. was about $13\frac{1}{2}°$ to the W. of geographical N.

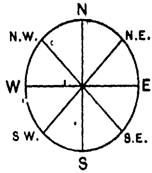


Fig. 59.

The student is probably familiar with the 8 chief points of the compass (Fig. 59), and they represent directions which vary 45° from one another.

In all there are 32 points of the compass, and the angles between consecutive directions are each $\frac{360}{32}°$ or $11\frac{1}{4}°$.

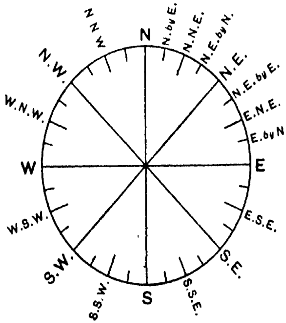


Fig. 60.

They are shown in Fig. 60.

Directions other than those on the compass may be described as:

N. 13° E.,	which means	13°	on the	E. side of N.,
N. 15° W.,	,,	15°	,,	W. ,, ,,
S. 60° E.,	,,	60°	,,	E. side of S.,
S. 75° W.,	,,	75°	,,	W. ,, ,,

35. The **Bearing** of an object is the number of degrees, measured in an Easterly or clockwise direction, between

a line drawn N. from the position of the observer and a line drawn in the direction of the object.

If the angle is measured from the geographical N., then the bearing is called a *true bearing*, and if measured from the magnetic north it is called a *magnetic bearing*.

Bearings of 32°, 125°, 220° and 340° are shown in the following diagrams :

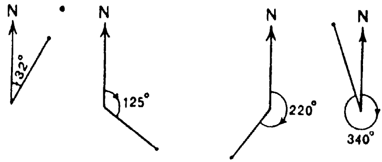


Fig. 61

EXERCISE 7.

1. In the given diagram measure the bearings of the objects A, B, C, D, E, F, when observed from O

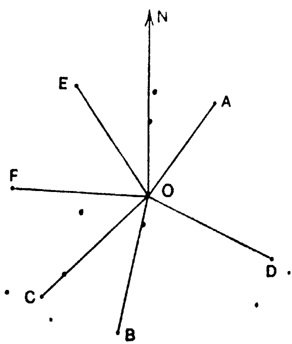


Fig. 62.

2. Express as bearings the following geographical directions : S.E., N.W., S.W., S.S.E., E.S.E., N.N.W., W.S.W., N.N.E., S. 50° E., S. 40° W., N. 30° E., N. 48° W.

[*To be solved graphically.*]

3. Give bearings of the successive paths of a patrol which travelled from (i) A to B, B to C, and C to D; state also the bearing of D from A; (ii) A to E, E to C, C to B. Give the bearing of E from B. Suppose the edge of the page runs N. and S.

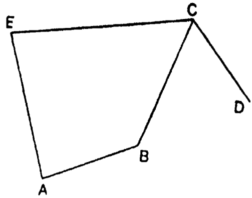


FIG. 63.

4. If A is 3 miles W. of O and B is 4 miles N. of A, find the distance and bearing of B from O.

5. B is 4·3 miles E. of A, and A is 3·2 miles S of O. Find the bearing of B from O.

6. If a man walks 3 miles S.E., 4 miles W. and 2 miles S.W., find from a diagram the bearing of his final position from the starting place.

7. A ship sails from a harbour 4 miles in a direction N.W., 2 miles N.E. and 3 miles E., find the bearing of the final position from the harbour.

8. Two scouts left a place A, one proceeded 2000 yds. on a bearing 110°, while the other went 3200 yds. on a bearing 190°. What would then be the bearing of each scout from the other, and what distance would they be apart ?

9. From A to B the bearing is 215°, the distance 500 yds.,
B to C ,, ,, 170°, ,, 450 yds.,
C to D ,, ,, 125°, ,, 360 yds.
Find the length and bearing of AD.

10. From A to B the bearing is 120°, the distance 400 yds.,
B to C ,, ,, 175°, ,, 340 yds.,
C to D ,, ,, 212°, ,, 370 yds.;
find the bearing of D from A and B respectively, and the distances AD, BD.

* **11.** The bearing of B from A is 115°, AB = 100 yds. The bearing of a fort F from A is 54°, and from B 340°. Find the distance of F from C, the middle point of AB.

12. Construct a triangle ABC in which BC = 500 yds., the bearing of C from B 290°, the bearing of A from B 340°, and the bearing of A from C 40°. Find a point D, the bearings of which from B and C are 220° and 150° respectively. What is the distance AD ?

DIAGRAMS DRAWN TO SCALE.

36. In representing the shape of a figure on paper, we have already seen that the actual lengths of the lines in the figure are drawn on paper by shorter lines; if every 1 foot (let us say) of the figure is represented by 1 cm., we say that the **scale** is 1 cm. to 1 ft.

Two such figures are said to be *similar*; they are of the same shape, and any angle in one figure is equal to the corresponding angle in the other.

The terms horizontal and vertical have already been explained on page 6.

If O is the position of an observer's eye, P an object at which he is looking, OA a horizontal line in the same vertical plane as OP, then

angle AOP is called the **angle of elevation** of the object when P is above O;

angle AOP is called the **angle of depression** when P is below O.

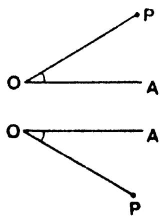


FIG. 64.

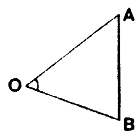


FIG. 65.

If the extremities A and B of an object AB (Fig. 65) are joined to the position of the observer's eye at O, then AB is said to **subtend** the angle AOB at the point O.

Ex. 1. *Draw a diagram on a scale of 1 cm. to 2 miles to represent the fact that Tewkesbury is 8 miles N.W. of Cheltenham and Gloucester 8 miles S.W. of Cheltenham. From the diagram measure the distance of Tewkesbury from Gloucester.*

Draw any line ACB to represent a line running N. and S., and let C be the position of Cheltenham. Make the angle

ACT $=45°$ and measure CT $=4$ cms.; then T will represent the position of Tewkesbury. Make the angle BCG $=45°$ and measure CG $=4$ cms.; then G will represent the position of Gloucester. Join TG.

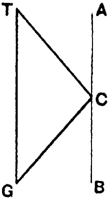


Fig. 66.

By measurement TG $=5{\cdot}65$ cms.

$\therefore$ distance of Tewkesbury from Gloucester is 11·3 miles.

Ex. 2. *A man observes the top of a spire* A *from a point* B *on the ground and finds that the angle of elevation is 19°. On walking 30 yds. towards the spire the angle is 25°. Find the height of the spire.*

Take a scale of 1 cm. to 20 yds.

Draw BC to represent 30 yds., *i.e.* of length 1·5 cms., and

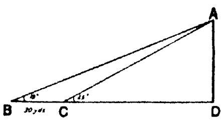


Fig. 67.

make $\hat{CBA}=19°$. Produce BC and make $\hat{DCA}=25°$. Let BA and CA meet at A, the top of the spire.

Draw AD perpendicular to BC produced; then by measurement AD $=2{\cdot}0$ cms.

$\therefore$ height of spire is 40 yds.

EXERCISE 8.

1. A town A is 7 miles due W. of B, while C is 6 miles N.E. of A; find the distance of C from B. (*Scale* 1 *cm. to* 1 *mile.*)

2. If A is 14·5 miles due N. of B and C is 10·4 miles due E. of A, find the distance and bearing of C when seen from B.
(*Scale* 1 *cm. to* 1 *mile.*)

3. A, B, and C are three villages. If AB $=13$ miles, BC $=10\frac{1}{2}$ miles, and it is found that the angle ABC $=55°$, find AC.
(*Scale* 1 *cm. to* 1 *mile.*)

4. From my house A it is possible to see two monuments B and C; B is in a N.E. direction and 350 yards away, while C is in a

S.E. direction, and 400 yards away. What is the distance **BC** ? (*Scale* 1 *in. to* 100 *yds.*)

5. A ladder 25 feet long is placed up against a wall so that its foot is 12 feet from the wall What is the inclination of the ladder to the vertical ? (*Scale* 1 *in to* 10 *ft.*)

6. The elevation of the top of a tower 520 feet away is 32°, find the height of the tower (*Scale* 1 *cm to* 100 *ft*)

7. If **B** is 5·5 miles N. of **A**, **C** 4·3 miles E of **B**, and **D** 3·3 miles N of **C**, find the distance of **D** from **A** (*Scale* 1 *cm to* 1 *mile*)

8. The top end of a string 12 feet long is fixed, and to the lower end is attached a weight If the string is displaced through an angle of 30°, what will then be the depth of the weight below the fixed end of the string ? (*Scale* 1 *cm to* 2 *ft*)

9. The string attached to a kite is 250 feet long, and makes an angle of 38° with the ground, what is the height of the kite ? (*Scale* 1 *cm to* 25 *ft*)

10. If a building 35 feet high subtends an angle of 55° at a point on the ground, what is the distance of the point from the building ? (*Scale* 1 *in to* 25 *ft*)

In the following examples the student should choose suitable scales, the actual scale taken being written under the diagram

11. A ship sails in a certain direction $5\frac{1}{4}$ miles, and then turns through an angle of 52 to the left, and sails another $3\frac{1}{2}$ miles Find the distance of its final from its original position

12. **C** is 18·2 miles W of **A** and **B** is 16 miles in a direction 42° N of W of **A** Find the distance and bearing of **C** as seen from **B**.

13. **X**, **Y**, **Z** are three towns, such that **XY** = 19 miles, **YZ** = 10·8 miles and **ZX** = 14·3 miles Find the distance of **X** from the line joining **YZ**.

14. A fixed lightship **B** bears N 20° W when observed from another ship **A**, when **A** sails due W for 5 miles, **B** is then due N. Find the distance of **B** from **A** at the first point of observation.

15. If the height of a tower is 120 ft and the angle of elevation from a point on the ground is 35°, find the distance of the point from the tower.

16. A man on the edge of a cliff observes that the angle of depression of a boat at sea is 25°. If the height of the cliff is 250 feet, what is the distance of the boat ?

17. If the shadow cast by a tower is 350 feet long when the altitude of the sun is 35°, what is the height of the tower ?

18. A man wishes to find the breadth of a river, so observes an object **A** on the opposite bank immediately opposite his own position **B**. He walks 120 yards down the bank to **C**, and on measuring the angle **BCA** finds that it is 36° ; find the breadth of the river.

19. B and C are two points on the straight bank of a river, such that BC = 65 yds. A is a point on the opposite bank, such that $A\hat{B}C = 40°$ and $A\hat{C}B = 35°$. What is the width of the river ?

20. From two points on the immediately opposite sides of a tree, the angles of elevation of the top are found to be 55° and 25°. If the two points are 120 feet apart, what is the height of the tree ?

21. A man starts from a point A and walks 5 miles due S. to a point B ; here he turns W. and walks to a point C, which is S.W. of A ; he now returns direct to A. What is the total distance he has walked ?

22. B and C are two points on a straight coast line running from W. to E. and 800 yards apart. A is a lighthouse which is N.E. of C and 20° N. of E. when seen from B. Find the distance of the lighthouse from each of the points B and C.

23. What will be the breadth of the zone of danger if a gun is fired from the top of a cliff 150 feet high and is depressed through angles varying from 10° to 15°, assuming that the projectiles travel in straight lines ?

24. The angles of elevation of the top of a flagstaff from two points distant 100 ft. from one another, in a vertical plane containing the flagstaff, are 40° and 25° ; what is the height of the flagstaff ?

25. A house is 80 feet high and the elevation of its top from a point on the ground is 75° ; the elevation of the bottom of a window from the same point is 32°. What is the height of the bottom of the window above the ground ?

26. An object on the shore bears from a ship N. 32° E., and after the ship has sailed for 7 miles in a direction N. 28° W., the object bears N. 45° E. Find the distance of the object from the ship at the second point of observation.

27. Two ships A and B drop their anchors at a distance 300 yards from one another, B bearing N. 32° W from A. A buoy is due E. of B and N. 40° E. of A. Find the distance of each ship from the buoy.

28. A boat leaves a lighthouse at the end of a harbour and sails N. 20° E. for 1000 yards, then N. 16° W. for 550 yards, and finally due W. for 400 yards. What is now its distance and bearing from the lighthouse ?

29. A tunnel is bored directly under a road ; if the road slopes up at an angle of 15° with the horizontal and the tunnel slopes down at an angle of 8° with the horizontal, how far will a point, which is 40 yards along the tunnel, be vertically under the road ?

30. When viewed from the top of a lighthouse, the depression of a ship at sea is 48°, and from a landing 20 feet lower down the depression is 40°. What is the distance of the ship from the bottom of the lighthouse ?

DEFINITIONS.

Axioms are usually spoken of as self-evident truths, they are such that no reasonable people deny or dispute them. They are taken for granted in ordinary life; for instance, if a carpenter has two exactly equal poles and cuts 6 inches from each of them, he is quite certain that the shortened poles are equal. He accepts the axiom that if equals be taken from equals the remainders are equal.

(i) Things which are equal to the same thing are equal to one another.

(ii) If equals be added to or subtracted from equals, the results are equal.

These two axioms include such statements as: the doubles of equal things are equal; the halves of equal things are equal. If one magnitude is greater than another and equals be added to both, the former sum is greater than the latter, and so on.

Postulates are claims that certain processes may be performed.

(i) Let it be granted a straight line (and only one) may be drawn through any two points.

(ii) Let it be granted that a straight line may be produced to any distance in either direction.

(iii) Let it be granted that a circle may be drawn with any point as centre and any length as radius.

Definitions. The following Definitions are placed here for reference; they should not be learnt *en bloc* but used as required when they are met with in the course of the book.

1. The **Volume** of a body is the amount of space it fills.

2. The **Surface** of a body is the boundary which separates it from the surrounding space.

3. Lines are the intersections of Surfaces.

The appearance of a **straight line** is well known and

(i) one straight line (and only one) can be drawn through two given points,

(ii) two straight lines cannot intersect in more than one point,

(iii) a straight line is the shortest distance between two points.

4. Points are the intersections of lines.

5. Angles.

We cannot here give a *precise* definition of the angle between two intersecting straight lines; it is sufficient for our purpose to understand how angles can be described and how they can be tested for equality.

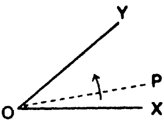


FIG. 68.

An angle is said to be described by the turning of a straight line about a point (called the angular point) in itself; and the amount of turning measures the magnitude of the angle.

Thus, if a straight line OP turns from OX to OY as in Fig. 68, it is said to describe the angle XOY.

OX and OY are called the *arms* (sometimes the legs) of the angle.

These arms can be of any lengths whatsoever.

N.B.—The angle is NOT the space XOY.

A **straight angle** is an angle the arms of which are in one straight line but on opposite sides of the angular point.

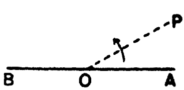


FIG. 69.

If AOB is a straight line and OP turns as in the Fig. 69, from OA to OB, then OP traces out a straight angle.

All straight angles are equal, for they can be made to coincide.

It is clear from Fig. 69, that as OP revolves, the angle AOP grows larger and BOP grows smaller.

At one position, and only one, such as OD, the two angles become equal; each is then said to be a **right angle**, and OD is said to be **perpendicular** to AB.

A **right angle** is one half of a straight angle, and consequently all right angles are equal.

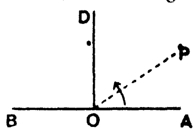


FIG. 70.

An **obtuse angle** is an angle greater than a right angle.

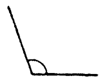


FIG. 71.

An **acute angle** is an angle less than a right angle.

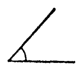


FIG. 72.

An angle which is larger than two right angles is called a **Reflex Angle.**

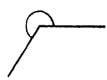


FIG. 73.

Degrees.

A right angle is too large for a convenient unit, so a smaller one, **a degree**, is chosen.

1 right angle = 90 degrees (written 90°),
1 degree = 60 minutes (written 60′).

A straight angle = 180°.

6. A **Rectilineal** figure is a figure bounded by straight lines.

7. A **Triangle** (△) is a plane figure bounded by three straight lines.

It has three angular points, any one of which may be called the **Vertex**, and the opposite side the **Base**.

8. An **Equilateral** triangle has all three sides equal.

9. An **Isosceles** triangle has two of its sides equal.

10. A **Scalene** triangle has all three sides unequal.

11. A **Right-angled** triangle has one right angle.

12. A **Circle** is a plane figure bounded by one curved line which is called the circumference, and is such that all points on the circumference are at the same distance from a certain point within the figure which is called the centre.

Sometimes the word circle is used for circumference; for instance, one reads, " two circles cannot cut each other at more than two points."

In Fig. 74, ARQBP is the circumference, C is the centre.

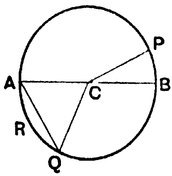


FIG. 74.

All the lines drawn from C to the circumference, such as CA, CQ, CB, CP, are equal; each one is called a **Radius.**

A line such as ACB, passing through the centre and terminated at both ends by the circumference, is called a **Diameter.**

Every diameter divides a circle into halves, each of which is called a **Semi-circle.**

A finite straight line such as AQ, joining any two points on the circumference, is called a **Chord.**

A part of the circumference between any two points, such as ARQ, is called an **Arc.**

The area, such as ARQ, contained between a chord and the arc it cuts off is called a **Segment.**

The area, such as CPB, contained between two radii and the arc between them is called a **Sector.**

13. A **Quadrilateral** is a plane figure bounded by four straight lines.

14. A **Parallelogram** is a four-sided figure with its opposite sides parallel.

15. A **Rectangle** is a parallelogram having a right angle.
It can be proved that if one angle is a right angle all the angles must be right angles.

16. A **Square** is a rectangle with two adjacent sides equal.
It can be proved that all the sides must be equal.

17. A **Rhombus** is a parallelogram with two adjacent sides equal but the angles not right angles.
It can be proved that all the sides must be equal.

18. A **Trapezium** is a quadrilateral with one pair of parallel sides.

19. A **Polygon** is a plane figure bounded by more than four straight lines.

20. A **Pentagon** is a five-sided polygon.

21. A **Hexagon** ,, six-sided ,,

22. A **Heptagon** ,, seven-sided ,,

23. An **Octagon** ,, eight-sided ,,

24. A **Regular Polygon** has all its sides equal, and all its angles equal.

The following abbreviations may be used :

$\therefore$	for	therefore,	$\angle$ or $\wedge$	for	angle,
$\because$	,,	because,	rt. $\angle$	,,	right angle,
$=$	,,	is equal to,	perp. or $\perp^{ar}$	,,	perpendicular,
$\equiv$	,,	is congruent with,	$\triangle$	,,	triangle,
$\parallel$	,,	parallel or is parallel to,	par^{m}. or $▱^{m}$	,,	parallelogram,
			sq.	,,	square,
$>$	,,	is greater than,	rectil.	,,	rectilineal,
$<$	,,	is less than,	rect.	,,	rectangle,
pt.	,,	point,	$\odot$	,,	circle,
str.	,,	straight,		,,	difference between.

VERIFICATION OF IMPORTANT GEOMETRICAL FACTS.

FACT 1.

Draw an angle $X\hat{O}P$ of 50° say, using the protractor; produce XO to some point X'; measure $X'\hat{O}P$ and add the two angles together.

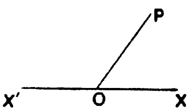


FIG. 75.

Repeat this six times, varying the size of the angle $X\hat{O}P$ each time.

In every case the sum of the two angles, which are called **Adjacent Angles**, should be 180°.

Having seen that the sum is 180° in all the cases you have drawn, you will not be surprised to learn the Geometrical Fact, that in any and every figure, such as Fig. 75,

$$X\hat{O}P + P\hat{O}X' \text{ is bound to equal } 180^\circ.$$

This Geometrical Fact is stated thus:

If one straight line meets another the sum of the 2 adjacent angles must be 2 right angles (or 180°).

Also note that if a line rotate from the position OX through OP to OX', the angle turned through is 2 right angles; but this angle may be described by first turning through the acute angle XOP and afterwards through the obtuse angle POX'.

Thus $\qquad X\hat{O}P + P\hat{O}X' = 2$ rt. angles.

The two angles XOP and POX', which together make up 180°, are said to be **Supplementary**, and either angle is said to be the **Supplement** of the other.

It should also be noticed that the drawing has not **proved** the Fact; it has *verified* it in six cases, and has thus suggested that the Fact stated is probably true.

COR. 1.

$$\widehat{AOK_1} + \widehat{K_1OK_2} + \widehat{K_2OK_3} + \widehat{K_3OK_4} + \widehat{K_4OB} = 180^\circ \text{ (Fig. 76).}$$

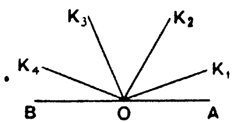


FIG. 76.

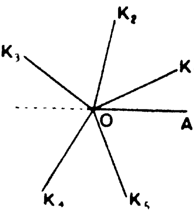


FIG. 77.

COR. 2.

$$\widehat{AOK_1} + \widehat{K_1OK_2} + \widehat{K_2OK_3} + \widehat{K_3OK_4} + \widehat{K_4OK_5} + \widehat{K_5OA} = 4 \text{ right angles} = 360^\circ \text{ (Fig. 77).}$$

FACT 2.

Think of any two angles which are supplementary, say 42° and 138°.

From a point O in a straight line OA make $\widehat{AOB} = 42^\circ$, and on the opposite side of OA make $\widehat{AOC} = 138^\circ$. On testing

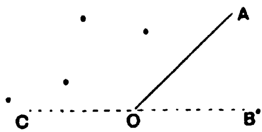


FIG. 78.

with the edge of a ruler it will be found that BOC is a straight line.

Try the same again with angles 20° and 160°; 60° and 120°; 80° and 100°.

Then try non-supplementary angles 45° and 130° ; 70° and 120° ; 30° and 160° ; and test these cases with the edge of a ruler.

You will now credit the Geometrical Fact :

If at a point in a straight line, two other straight lines, on opposite sides of it, make the sum of the adjacent angles equal to two right angles, these two other straight lines must be in one and the same straight line.

COR. If three (or more) angles

$$AOB + BOC + COD = 180°,$$

then DOA must be a straight line.

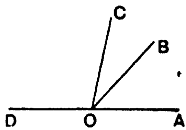


FIG. 79.

FACT 3.

Draw six pairs of straight lines intersecting each other, such as AOH and BOK.

Measure the angles AOB, AOK, KOH, HOB ; in every case it will be found that $A\hat{O}B = K\hat{O}H$ and $K\hat{O}A = B\hat{O}H$.

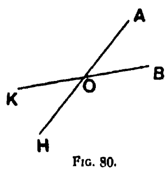


FIG. 80.

The Geometrical Fact is :

If two straight lines intersect, the vertically opposite angles must be equal.

It may be noticed that if the line KOB rotates round O into the position HOA, then the arms OB and OK, in moving into their new positions OA and OH, obviously rotate through equal angles AOB and KOH.

EXERCISE 9.

1. In Fig. 81 there are 6 pairs of equal angles; state which they are, and give a reason.

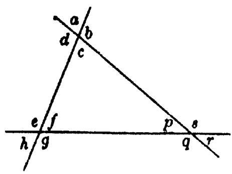


FIG. 81

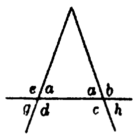


FIG. 82.

2. In Fig 82 the angles marked a are equal. Somebody states that $\hat{b} = \hat{e}$; that $\hat{g}$ is greater than $\hat{h}$; that $\hat{c}$ is less than $\hat{d}$.

Are any of these statements true? Give precise reasons for your answer in each case.

3. In Fig 83 a student is set to measure the angle a, and gives as his result 140°.

Is he correct? Do not measure, but simply give a reason for your answer.

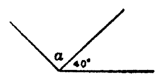


FIG. 83.

4. Draw an angle of 40°; produce one arm backwards and measure the angle thus formed.

Repeat for angles of 20°, 66°, 84°.

5. Draw a straight line and take any point in it; from the point draw two dotted lines on opposite sides of the first line, making angles of 62° and 116° with it.

The dotted lines will be nearly in one straight line, but not quite. Write an explanation of this.

6. Draw an angle of 70°; produce both arms backwards; measure and write down the three other angles now formed.

Repeat for an angle of 100°.

7. The line PQ is known to bisect the angle AOC; can you give any reason for determining whether it will or will not bisect $D\hat{O}B$? (Fig. 84).

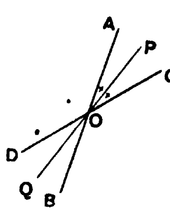


FIG. 84.

8. What are the supplements of the angles 2°, 100° 39°, 28°, 90°, 60°, 45°?

9. With Fig. 85, give the following values to the angles CBD and DBA; in which cases will CBA be a straight line?

(*a*) 152°, 28°, (*b*) 143°, 42°; (*c*) 117°, 62°,

(*d*) 133°, 57°, (*e*) 128°, 52°.

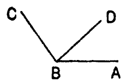


Fig. 85

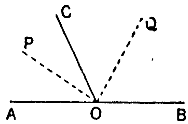


Fig. 86.

10. Angle AOC = 62°, COB = 118°, and OP, OQ bisect angles AOC, COB respectively. Write down the values of the angles POC, COQ, POQ. Thence prove generally that the bisectors of the adjacent angles which one straight line makes with another are at right angles (Fig. 86).

11. Prove that the bisectors of two vertically opposite angles are in one straight line.

12. ABC is a triangle having the angle ABC equal to the angle ACB, and the sides AB and AC are produced to D and E respectively. Prove that the angle DBC is equal to the angle ECB.

13. A, B, C are three points in a straight line. AP and BQ are drawn so that $P\hat{A}C = Q\hat{B}C$. Prove that

$$P\hat{A}C + Q\hat{B}A = \text{two right angles}$$

14. ABC is a triangle having the side BC produced to D; it is given that the angle ACD is equal to the sum of the angles BAC and ABC. Prove that the three angles of the triangle are together equal to two right angles.

Parallel straight lines are defined to be straight lines, in the same plane, which do not meet, however far they are produced in either direction.

Let PA_1 be a line passing through a given pt. P and intersecting a given straight line XY in A_1.

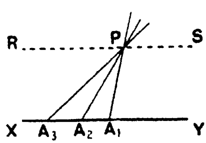


Fig. 87.

As PA_1 rotates about P in a clockwise direction, the angle PAY becomes smaller and smaller, the point of intersection A moves farther and farther away from Y, and PA approaches more and more nearly to the direction of YX.

It will probably be accepted as obvious that PA eventually reaches a position PR in which, however far it is produced either way, it does not meet XY.

This limiting position RPS is a line parallel to XY.

If the rotation is continued beyond this *one* position, then the line AP produced will meet XY produced towards the right.

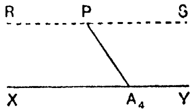


FIG. 88.

This result is enunciated in *Playfair's Axiom*.

Through a given point one straight line, and only one, can be drawn parallel to a given straight line.

FACT 4.

Draw $\hat{AOB}$ of any magnitude, say 40°. Produce OA to Q; make $\hat{QAC}$ also equal to 40°.

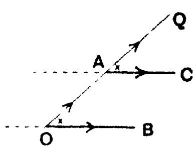


FIG. 89.

Then produce CA and BO as much as possible, and notice that they never appear to approach each other, *i.e.* they are parallel.

Do this six times, varying the angle, and produce the lines as far as possible. You will notice these lines show no appearance of being likely to meet.

The equal angles QAC, AOB are called **Corresponding Angles.**

The Geometrical Fact is:

If a straight line meets two straight lines and makes the corresponding angles equal, the two straight lines are parallel.

Suppose two men walk along OAQ; and at O one man turns 40° to the right along OB, and at A the other turns 40° to the right along AC. They will then be walking along ***Parallel*** lines OB, AC.

FACT 5.

Draw $A\hat{O}B$ of any magnitude, say 50°. Make at A the angle CAO also 50°.

Then produce the lines BO and CA as far as you can, and notice that they never appear to approach each other, *i.e.* that they are in the same direction or parallel.

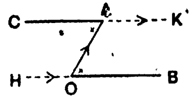


FIG. 90.

The equal angles CAO and AOB, on opposite sides of AO, are called **Alternate** angles.

The Geometrical Fact is :

If a straight line meets two straight lines and makes the alternate angles equal, the straight lines must be parallel.

A man walking along HO might turn 50° to the left at O and then walk along OA ; if at A he turns 50° to the right, he will then be walking along a line AK parallel to HO.

FACT 6.

Draw $A\hat{O}B$ of any magnitude, say 35°. Make $O\hat{A}C$ the supplement, viz. 145°.

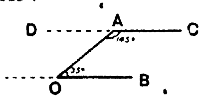


FIG. 91.

Then produce the lines CA and BO as much as possible.

Do this six times, varying the angle, and notice again how the lines never approach each other.

The Geometrical Fact is :

If a straight line meets two straight lines and makes the sum of the two interior angles on one side of the line equal to two right angles, the two straight lines must be parallel.

Also, since $\hat{DAO}$ (the supplement of $\hat{OAC}$) $= 180° - 145°$
$= 35°$
$= \hat{AOB}$,

∴ the alternate angles DAO and AOB are equal. Fact 5 tells us the lines are parallel.

FACT 7.

Draw any pair of parallel lines, by moving the set square along the edge of the ruler as in Fig. 92.

The two parallel lines AB and A′B′ should be produced as much as possible.

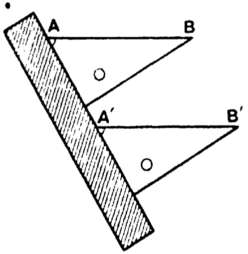


FIG. 92.

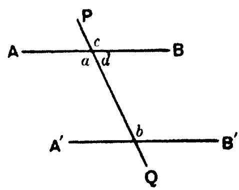


FIG. 93

Fact 4 tells us the lines are parallel, $\hat{A}$ being equal to $\hat{A'}$.
Draw any cross line PQ as in Fig. 93.

Measure the angles a, b, c, d.

Notice $\hat{c} = \hat{b}$, $\hat{a} = \hat{b}$, $\hat{b} + \hat{d} = 180°$.

Do this six times, varying the slope of PQ.

The Geometrical Fact is :

If a straight line cuts two parallel straight lines,

(i) the corresponding angles are equal,

(ii) the alternate angles are equal,

(iii) the sum of the two interior angles on the same side of the line must be two right angles.

We may deduce these results from Playfair's Axiom.
Let MTRN cut the two parallel lines AB and CD.
Then we have to prove that

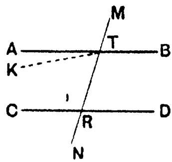


FIG. 94.

(i) $\widehat{ATR} = \widehat{TRD}$ and $\widehat{BTR} = \widehat{TRC}$,

(ii) $\widehat{MTB} = \widehat{TRD}$,

(iii) $\widehat{BTR} + \widehat{TRD} = 180°$.

Proof. (i) Suppose $\widehat{ATR}$ is not equal to $\widehat{TRD}$, let KT be the line which makes $\widehat{KTR}$ = alternate $\widehat{TRD}$.

Then by Fact 5, KT must be $\parallel$ to CD,

$\therefore$ AT and KT are both $\parallel$ to CD, which is contrary to Playfair's Axiom;

$$\therefore \widehat{ATR} = \widehat{TRD}.$$

(ii) Now $\widehat{ATR}$ = the vertically opposite $\widehat{MTB}$;

$$\therefore \widehat{MTB} = \widehat{TRD}$$

(iii) Also $\widehat{ATR} + \widehat{BTR} = 180°$,

$$\therefore \widehat{TRD} + \widehat{BTR} = 180°.$$

COR. **Straight lines which are parallel to the same straight line are parallel to one another.**

Suppose the lines AB and CD are each parallel to XY.

Draw EH cutting the lines at F, G, H.

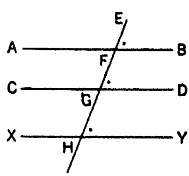


FIG. 95.

$\because$ AB is parallel to XY,

$\widehat{EFB}$ = corresponding $\widehat{GHY}$.

$\because$ CD is parallel to XY,

$\widehat{FGD}$ = corresponding $\widehat{GHY}$.

$\therefore$ $\widehat{EFB}$ = corresponding $\widehat{FGD}$; $\therefore$ AB is parallel to CD.

EXERCISE 10.

1. Given one angle 58°, find all the other angles if P and Q are parallel (Fig. 96).

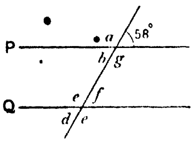


Fig. 96

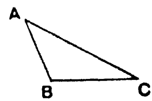


Fig. 97.

2. In Fig. 97, somebody measures the angles thus, B = 145°, C = 35°. Without measuring, state whether he is correct or not. Give reasons.

3. Two men start walking from the same point, the first due North and the second due East; the latter, after a while, turns 30° to the North, and later 60° to the North. What is now the connection between their directions? Indicate by a sketch.

4. Draw pairs of parallels as in Fig. 98, and give all the angles which are equal, with reasons.

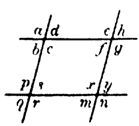


Fig. 98

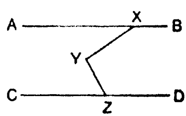


Fig. 99.

5. If the lines AB and CD are parallel, and $\hat{AXY} = 32°$, $\hat{YZC} = 44°$, what is the angle XYZ? (Fig. 99).

6. Find the angle XYZ if $\hat{BXY} = 54°$, $\hat{DZY} = 38°$, and AB, CD are parallel (Fig. 100).

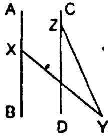


Fig. 100.

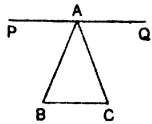


Fig. 101.

7. PQ is parallel to BC. If $\hat{ABC} = 62°$, $\hat{ACB} = 68°$, find the three angles BAP, BAC, CAQ (Fig. 101).

8. If a straight line is perpendicular to one of two parallel straight lines, it is perpendicular to the other.

9. If one pair of straight lines AB, BC, are respectively parallel to another pair of straight lines, DE, EF, the acute angle between one pair must be equal to the acute angle between the other pair (Fig. 102).

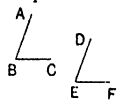


FIG. 102.

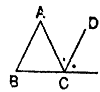


FIG. 103.

10. If the bisector CD of an exterior angle of a triangle is parallel to the opposite side BA, the triangle must have two of its angles equal (Fig. 103).

11. Through C a line is drawn making $T\hat{C}A = \hat{A}$; is there any relation between the lines AB and TC? Give reasons for your answer. Why is $T\hat{C}D = A\hat{B}C$? (Fig. 104).

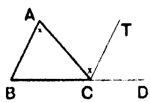


FIG. 104.

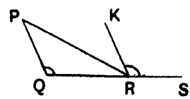


FIG. 105.

12. Through R (in Fig. 105), a line is drawn making $K\hat{R}S = \hat{Q}$; is there any relation between the lines PQ and KR? Give reasons.

13. In Fig. 106 the angles are equal as marked. What do you know about AB and DC, and about AD and BC?

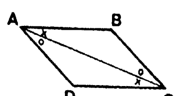


FIG. 106.

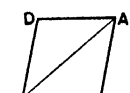


FIG. 107.

14. Draw pairs of parallels as in Fig. 107; join AB, and mention two pairs of equal angles.

Do you think **Parallelogram** a good name for the fig. ACBD? Give a reason.

15. If the opposite sides of a quadrilateral are parallel, and if one angle is a right angle, all the angles must be right angles.

16. If the opposite sides of a quadrilateral are parallel, the opposite angles must be equal.

17. In Fig. 108, XAY is parallel to BC; $\widehat{ABC} = 42°$, $\widehat{ACB} = 63°$. Find the angles PAX, XAB, QAY, YAC.

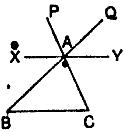


FIG. 108.

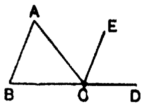


FIG. 109.

18. In Fig. 109, if $\widehat{BAC} = 63°$, $\widehat{ACB} = 47°$, $\widehat{ECD} = 70°$, prove that CE ∥ BA.

19. Draw an angle BAC = 60° and make AB = AC = 5 cms. Through B and C draw lines BD and CD parallel to AC and AB respectively. Join AD, BC, cutting in E. Measure the four angles at E and the lengths of BC, AD. The figure ABCD is called a **Rhombus**.

By means of Fact 5 we have a method of *drawing through a given point a line parallel to a given line,* without using a set square.

Suppose it is required to draw through P a line parallel to XY.

Join P to any point A on the line.

By the method given on page 20 make the angle APB = angle PAY; produce BP to C.

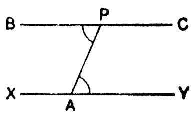


FIG. 110.

Then because $\widehat{APB}$ = alt. $\widehat{PAY}$, it follows that BC ∥ XY.

Draw a line PQ, and along it measure any number of equal distances PA, AB, BC.

By means of a set square and ruler, draw parallel lines through A, B, C.

Through P draw any straight line cutting these parallel lines in R, S, T.

By measuring, it will be found that

PR = RS = ST.

This suggests a method of ***dividing a finite straight line into any number of equal parts.***

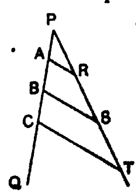


FIG. 111.

Suppose it is required to divide AB into 4 equal parts.

Through A draw AX at any angle.

Along AX mark off 4 equal parts AC, CD, DE, EF, of any convenient length.

Join BF.

Through C, D, E, draw lines CG, DH, EK $\parallel$ FB.

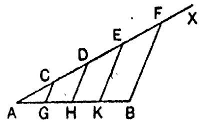


FIG. 112

Then AG, GH, HK, KB are equal to one another.

(For Proof, see Theorem 15.)

EXERCISE 11.

1. Draw a straight line 11·5 cms long, and by construction divide it into 3 equal parts Measure the lengths of the three parts.

2. Divide a straight line 3·7 inches in length into 7 equal parts by a geometrical construction, and measure the length of a straight line AB made up of the sum of three of these divisions

3. Draw a line AB 3½ inches in length, at A and B erect perpendiculars AD, BC each equal to 3½ inches, join DC The figure ABCD is a **square** By means of a geometrical construction divide AB and DC each into three equal parts at the points X, Y, P, Q.

Join QX, and measure its length

4. Draw a triangle ABC so that AB 3½ ins, AC = 2 ins, BC = 4 ins. Through C and B draw CD, BD parallel to AB, AC respectively.

By a geometrical construction divide AB into 3 equal parts at E and F. Join DE, and measure its length.

5. Draw an equilateral triangle ABC each side 5 cms. long. By a geometrical construction trisect AB at P, Q. Thence trisect AC, CB at R, S and T, V respectively (Fig 113).

Join PR, ST, VQ The figure QPRSTV is called a **hexagon.**

Measure PR, ST, VQ.

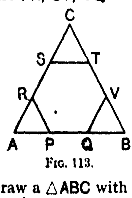


FIG. 113.

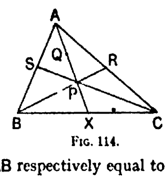


FIG. 114.

6. Draw a $\triangle$ABC with BC, CA, AB respectively equal to 5·4, 4·6, 3·8 cms. (Fig. 114).

By geometrical constructions bisect BC at X, join XA, and trisect XA at P, Q.

Join BP, CP, and produce to R, S.

Measure AR, RC, AS, SB.

7. Draw a line AC 7·5 cms. long, and divide it into 7 equal parts by a geometrical construction (Fig 115).

Construct a four-sided figure ABCD, of which the above line is one diagonal from the following data :

BD cuts AC at O, so that AO is three-sevenths of AC, AB = 4·4 cms, $\hat{AOB}$ = 47°, $\hat{BCD}$ = 59°. Measure BD.

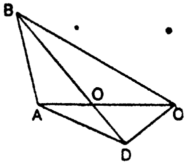


Fig. 115.

8. Construct a quadrilateral (a four-sided figure) ABCD from the following data. AB = 5·2 cms, AD 10·8 cms, $\hat{BAD}$ = 80°, and the diagonals AC, BD are equal and include an angle of 64° Measure CD

ANGLES OF A TRIANGLE AND POLYGON.

(i) Let a man start walking from A to B, and on arriving at B turn along the direction BC ; he will thus turn through the angle marked b.

On arriving at C, let him turn along CA ; he will here turn through the angle marked c.

On arriving at A, let him turn along the original line AB ; he will here turn through the angle marked a.

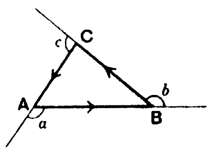


Fig 116.

Since he is now facing in his original direction, and he has been turning to the left at each corner, it follows that he has *turned through* 4 *right angles* ;

$$\therefore\ a+b+c=4 \text{ right angles.}$$

$\therefore$ by Fact 1,

$$(2 \text{ rt. angles} - \hat{CAB}) + (2 \text{ rt. angles} - \hat{ABC}) + (2 \text{ rt. angles} - \hat{BCA}) = 4 \text{ rt. angles};$$

$$\therefore\ \hat{CAB} + \hat{ABC} + \hat{BCA} = 2 \text{ rt. angles.}$$

(ii) If the angles A, B, C are measured by a protractor, we find that their sum is 180°.

The Geometrical Fact is :

The sum of the three angles of any triangle must be two right angles or 180°.

We shall now give a more formal proof of this result.

THEOREM 1.

The sum of the three angles of any triangle must be two right angles.

Let ABC be a triangle.

We have to prove that

$$\hat{A} + \hat{B} + C = 180°.$$

Construction. Produce BC to D, and through C draw CK || BA.

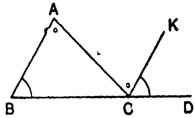


FIG. 117.

Proof. Since CK || BA,

$$\therefore\ \widehat{KCA} = \text{alt. } \widehat{BAC},$$

and $$\widehat{KCD} = \text{corr. } \widehat{ABC}.$$

$$\therefore\ \widehat{BAC} + \widehat{ABC} + \widehat{BCA} = \widehat{KCA} + \widehat{KCD} + \widehat{ACB}$$
$$= 180°, \text{ since BCD is a straight line.}$$

COR. 1. *If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the interior opposite angles.*

From the above proof we see that

$$\hat{A} + \hat{B} = \widehat{KCA} + \widehat{KCD} = \widehat{ACD};$$

hence the exterior angle of a $\triangle$ is $>$ either of the interior opposite angles.

COR. 2. *If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angles are also equal.*

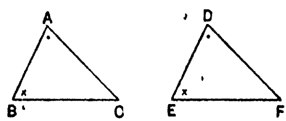


FIG. 118.

If $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$, then

$$\hat{F} = 180° - \hat{D} - \hat{E} = 180° - \hat{A} - \hat{B}$$
$$= \hat{C}.$$

COR. 3. *In a right-angled triangle* (*i.e.* a $\triangle$ having one angle 90°),

(i) *the right angle is the greatest angle.*

(ii) *the sum of the remaining angles is equal to a right angle.*

Since $\hat{A}+\hat{B}+\hat{C}=180°$ and $\hat{C}=90°$,

$$\therefore\ \hat{A}+\hat{B}=90°,$$

and consequently A and B are each $< 90°$.

If the sum of two angles, such as A and B, is one right angle, they are said to be **complementary**.

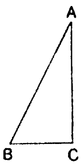


FIG. 119.

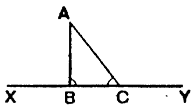


FIG. 120.

COR. 4. *Only one perp. can be drawn to a straight line from an external point.*

Suppose that both AB and AC are perps. from A to the line XY, then, since $A\hat{B}C + B\hat{C}A$ = two rt. angles, it would follow that the three angles of $\triangle$ABC would be greater than two rt. angles.

Since this result is impossible, it follows that AB and AC cannot both be perp. to XY.

EXERCISE 12.

1. If 2 angles of a triangle are 30° and 40°, find the third angle.
2. ,, ,, ,, 80° and 90° ,, ,,
3. ,, ,, ,, 52° and 74° ,, ,,
4. ,, ,, ,, 23° and 113° ,, ,,
5. ,, ,, ,, 40° and 50° ,, ,,
6. ,, ,, ,, 20° and 70° ,, ,,
7. ,, ,, ,, 30° and 60° ,, ,,
8. If in a right-angled triangle, one of the angles is 23°, what is the other acute angle?

9. If in a right-angled triangle one of the angles is $x°$, what is the other acute angle ?

10. If the angles of a triangle contain x, x, $2x$ degrees respectively, what is the value of x ?

11. If the three angles of a triangle are equal, how many degrees does each contain ?

12. If the smallest angle of a triangle contains 24° and the other two are equal, what is the size of each ?

13. An exterior angle of a $\triangle$ is 120°, and one of the interior angles is 20°. Find the other two angles of the triangle.

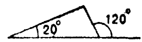


FIG. 121.

14. In the triangle ABC, which are the interior opposite angles to DAC, EAB, ABF, ACK, CBG, BCH, respectively ? (Fig. 122).

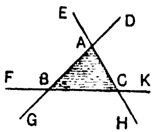


FIG. 122.

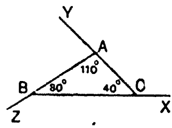


FIG. 123

15. The angles of a $\triangle$ABC being 30°, 40°, 110°, find the exterior angles ACX, CBZ, BAY (Fig. 123).

16. Draw any *four sided figure*, and by joining two of the opposite corners, find the sum of the four angles of the figure.

Prove that, if the opposite angles of a quadrilateral are equal, its opposite sides are parallel.

17. In a triangle ABC, CA is produced to D. AE, CE bisect angles BAD, ACB respectively (Fig. 124).

If the angles A, B, C are 72°, 66°, 42°, find the angle AEC.

[Note that angle DAE is exterior angle of $\triangle$AEC.]

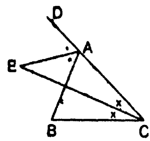


FIG. 124.

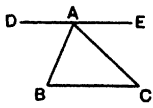


FIG. 125.

18. Draw any triangle ABC, and through A draw DAE parallel to BC. What is the connection between the angles DAB and ABC, and also between EAC and ACB ? From this result prove that the three angles of the triangle ABC equal two right angles (Fig. 125).

19. AO, BO, CO bisect the angles A, B, C of a triangle ABC. If A, B, C are equal to 68°, 64°, 48° respectively, find the values of the angles AOB, BOC, COA (Fig. 126).

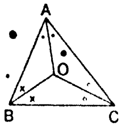


FIG. 126.

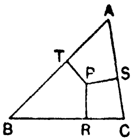


FIG. 127.

20. P is any point inside the triangle ABC and PR, PS, PT are perp. to BC, CA, AB respectively (Fig. 127).
If the angles SPT, RPT, SPR equal 124°, 138°, 98° respectively, find the values of the angles A, B, C. [See Qn 16.]

21. In the triangle ABC, BD bisects the angle B. If $\hat{A} = 67°$, $A\hat{D}B = 78°$, find the values of the angles B and C (Fig. 128).

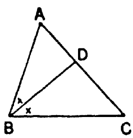


FIG. 128.

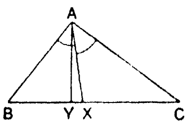


FIG. 129.

22. In the triangle ABC, AX bisects the angle A and AY is perp. to BC. If $\hat{B} = 50°$ and $\hat{C} = 34°$, find the values of the angles BAC, BAY, XAY, AXB.

23. Draw a straight line of length 3·5 centimetres; on it describe a triangle having one of the base angles three times and the other base angle six times the angle at the vertex. Measure the lengths of the sides in centimetres.

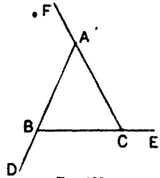


FIG. 130.

24. The sides AB, BC, CA of a triangle are produced in order to D, E, F respectively. The angle CBD = 110°, and the angle ACB = 75°. Calculate the angles BAF, ACE (Fig. 130).

A *convex polygon* is one in which each angle is less than two right angles.

THEOREM 2.

In a polygon of n sides, the sum of the interior angles is equal to (2n − 4) right angles.

Let ABCDEFG be any polygon with n sides.

Then we have to prove that

$$\hat{A} + \hat{B} + \hat{C} + \hat{D} + \ldots = (2n - 4) \text{ right angles.}$$

Construction. Take any point O inside the polygon and join OA, OB, OC, etc., forming n triangles OAB, OGA, etc.

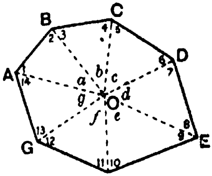


FIG. 131.

Proof. Since the sum of the angles of a triangle = 2 rt. angles,

∴ the sum of the angles of n triangles = $2n$ rt. angles,

i.e. all the angles 1, 2, 3, etc. + angles a, b, c, etc. = $2n$ rt. angles,

∴ all the angles 1, 2, 3, etc. + 4 rt. angles = $2n$ rt. angles;

∴ all the angles 1, 2, 3, etc. = $(2n - 4)$ rt. angles,

$$\textit{i.e. } \hat{A} + \hat{B} + \hat{C} + \ldots = (2n - 4) \text{ rt. angles.}$$

COR. *If the sides of any convex polygon are produced in order, the sum of all the exterior angles is four right angles.*

Suppose the polygon has n sides.

Since the angles $1 + a = 2 + b = \ldots$
$= 2$ rt. angles,

∴ angles 1, 2, 3, etc. + a, b, c, etc.
$= 2n$ rt. angles;

but angles a, b, c, etc.
$= (2n - 4)$ rt. angles;

∴ angles 1, 2, 3, etc.
$+ (2n - 4)$ rt. angles
$= 2n$ rt. angles;

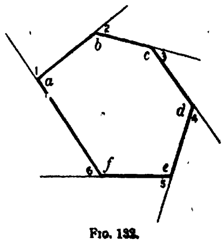


FIG. 132.

∴ sum of ext. angles 1, 2, 3, etc. = 4 rt. angles.

By a method similar to that in Section (i) on page 49, if a man walk completely round the polygon, he will turn through 4 rt. angles, and this turn will be made up of the separate turns 1, 2, ... 6.

The sum of all the angles in

a **Quadrilateral**	= (8 − 4) right angles = 4 right angles,
a **Pentagon**	= (10 − 4) right angles = 6 right angles,
a **Hexagon**	= (12 − 4) right angles = 8 right angles,
a **Heptagon**	= (14 − 4) right angles = 10 right angles,
an **Octagon**	= (16 − 4) right angles = 12 right angles,

and so on.

Thus, an angle in

a regular Quadrilateral	= $\frac{4}{4}$ = 1 right angle = 90°,
a regular Pentagon	= $\frac{6}{5}$ right angle = 108°,
a regular Hexagon	= $\frac{8}{6}$ right angle = 120°,
a regular Heptagon	= $\frac{10}{7}$ right angle = $128\frac{4}{7}$°,
a regular Octagon	= $\frac{12}{8}$ right angle = 135°.

EXERCISE 13.

1. Draw any quadrilateral, measure all the angles, and add them up.

2. Draw any pentagon (5-sided figure), measure all the angles, and add them up.

3. Draw any hexagon (6-sided figure), measure all the angles, and add them up.

4. Draw any octagon (8-sided figure), measure all the angles, and add them up.

5. If the angle of a regular polygon contains $\frac{14}{9}$ rt. angle, how many sides has it ?

6. If the angle of a regular polygon contains 150°, how many sides has it ?

7. Find the number of sides in a regular polygon given that each angle contains 144°.

8. The magnitudes of the angles of a quadrilateral are proportional to the numbers 3, 4, 5, 6. Find their magnitudes.

9. If five of the angles of a hexagon are 88°, 102°, 115°, 150°, 170° respectively, what is the magnitude of the sixth angle ?

10. The sum of 5 angles of a polygon is 650°, and the other angles are each 125°. Find the number of sides.

11. Draw any circle, mark off a chord equal to the radius; from *one end of this chord strike off another equal chord, and so on. Measure each angle in the figure you get.*

12. AB, BC, CD are three consecutive sides of a regular five-sided figure. AB and DC when produced meet at F. Calculate the number of degrees in the angle AFD and each angle of the triangle ABE.

Prove that BE is parallel to CD.

CONGRUENT TRIANGLES.

We have already worked out a number of problems in which it was necessary to construct a triangle, and we should have noticed that in all cases it was essential to know 3 elements of that triangle. We have constructed triangles in which we knew

(i) 2 *sides and the included angle* (Fact 8),
(ii) 2 *angles and* 1 *side* (Fact 9),
(iii) 3 *sides* (Fact 10);

and shall examine the further cases where we are given

(iv) 2 *sides and an angle opposite one side* (p. 73),
(v) 3 *angles* (p. 75).

If two triangles are equal in all respects, they are said to be **congruent.**

FACT 8.

Draw an angle $\widehat{PAQ} = 30°$. Take AH = 4 cms.; AK = 5 cms. Join HK.

You have now drawn a triangle AHK having two sides 4 and 5 cms. and the included angle 30°.

Try to draw another triangle BMD with these numbers, but of a different shape and size from AHK; you will find this impossible; do not accept the statement that it is

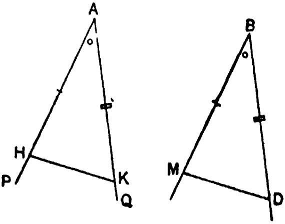


FIG. 133.

impossible without making an attempt; it is important that you should *convince yourself* of the impossibility.

In fact the triangle can be ***drawn in one way*** only from the given data.

The Geometrical Fact is:

If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, the triangles must be congruent.

FACT 9.

Draw a straight line AB 5 cms. Make $B\hat{A}C = 60°$ and $A\hat{B}C = 40°$. You have now described a triangle with two angles 60° and 40°, and the side *adjacent* to them 5 cms.

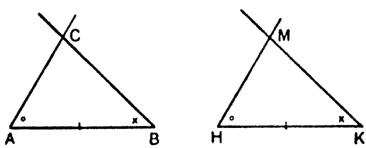


FIG. 134.

Draw a second $\triangle$ MHK with the same numbers as $\triangle$ CAB; the triangle is identical with the previous, since once more the triangle can be *drawn in one way* only with the given data.

Since we have seen that the sum of the three angles of a triangle is 180°, it follows that if two triangles have two angles of one equal to two angles of the other, each to each, then the third angles must be equal also. In Fig. 134, this means that $\hat{C} = \hat{M}$, and the sides AB, HK are *opposite* these equal angles.

The Geometrical Fact is:

If two angles and one side of a triangle are respectively equal to two angles and the corresponding side of another triangle, the triangles must be congruent.

EXERCISE 14.

1. Draw an angle of 40°; take a point 3·5 cms. along one arm and another point 2·5 cms. along the other arm. Join the two points. In the triangle so formed, measure the third side and the other two angles.

2. Draw a triangle with sides 8 and 4 cms., and having the included angle 62°. Measure the third side and the remaining angles.

3. Draw a line BC 4 cms. long; at B make an angle 60° and at C an angle 50°, and thus make a triangle; measure the other two sides.

4. Take the same question as the last, with the exception of making 60° at C and 50° at B. Has this made any difference to the triangle formed? Give reasons for your answer.

5. Draw a triangle ABC with the sides AB and AC each equal to 5·4 cms. and the angle BAC = 40°. Draw a line AD so that

$$B\hat{A}D = D\hat{A}C = 20°,$$

i.e. a line *bisecting* the angle BAC.

Measure each of the angles ABC, ACB, ADB, ADC. What connection do you observe between them?

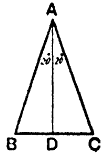


FIG. 135.

6. ABCD is a *square*, so that all its sides are equal and all its angles right angles. If E is the middle point of AD, prove that EB = EC. [Compare the $\triangle$ˢ EBA, ECD] (Fig. 136).

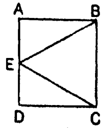


FIG. 136.

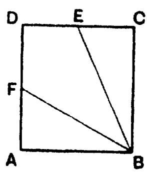


FIG. 137.

7. ABCD is a square (Fig. 137); if E and F are the middle points of CD, AD respectively, prove that

$$BE = BF \text{ and } E\hat{B}C = F\hat{B}A.$$

[Compare the $\triangle$ˢ BEC, BFA.]

8. Draw a line AB 4·4 cms. long, and construct the angles BAC, ABD each equal to 40°; make AC = BD = 2·6 cms.; join DA, CB.

Prove that AD = CB, $\hat{D} = \hat{C}$, $D\hat{A}C = C\hat{B}D$. Verify by measurement.

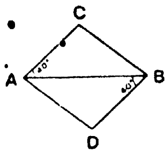


Fig 138.

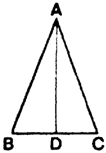


Fig. 139.

9. In the triangle ABC it is given that $\hat{B} = \hat{C}$, and that $B\hat{A}D = C\hat{A}D$. Prove that the triangle is isosceles and that AD bisects BC at right angles (Fig. 139)

10. In a quadrilateral ABCD, the diagonal AC bisects the angles BAD and BCD. Prove that AB = AD, CB = CD and $\hat{B} = \hat{D}$.

11. The triangles ABC and A′BC are on the same side of their common base BC, and the angle A′BC equals the angle ACB, and the angle A′CB equals the angle ABC, also AB and A′C intersect in O. Prove that the triangles AOC and A′OB are congruent.

12. ABCD is a *square*. Points P, Q are taken in the sides BA, BC respectively, such that BP = BQ. AQ, CP cut at O. Prove that AO = OC.

13. The bisector of an angle of a triangle is perpendicular to the opposite side. Prove that the triangle must have two of its sides equal.

14. A triangle has two equal angles; through the third angle a line is drawn perpendicular to the opposite side. Prove that this perpendicular must bisect the opposite side.

15. ABC is an isosceles triangle having AB = AC. If AD bisects the angle BAC and meets the base at D, prove that the triangles ABD and ACD are congruent, and that in particular $\hat{B} = \hat{C}$, and also BD = CD.

16. ABC is a triangle having $A\hat{B}C = A\hat{C}B$. X and Y are points in AB and AC respectively, such that BX = CY. Prove that X and Y are equidistant from the middle point of BC and that BY = CX.

17. **PQ is a straight line. PA and QB are drawn on the same side of PQ so that $A\hat{P}Q = B\hat{Q}P = 90°$, and AP = BQ. Prove that AQ = BP.**

18. **ABC is an isosceles triangle having AB = AC. P and Q are points in AB and AC such that AP = AQ. Prove that PC = QB.**

19. A triangle ABC is turned over about its side BC as a hinge. Prove that the line joining the new and old positions of A is bisected by BC (Fig. 140).

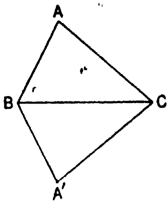


Fig. 140.

20. ABCD is a quadrilateral, and BD a diagonal. AB = CD and $A\hat{B}D = B\hat{D}C$. Prove that the triangles ABD and CDB are congruent, and that in particular $\hat{A} = \hat{C}$ and AD = BC.

21. ABCD is a quadrilateral with AC a diagonal; $B\hat{A}C = D\hat{C}A$; $B\hat{C}A = D\hat{A}C$. Prove that AB = CD.

22. PQRS is a quadrilateral such that the diagonal QS bisects the angles at Q and S. Prove that the diagonals PR and QS are at right angles.

23. ABC is a triangle having $\hat{B} = \hat{C}$, and the base BC is trisected at H, K; $B\hat{A}H = C\hat{A}K$. Prove that AH = AK.

24. Justify the following two methods of finding the breadth of a river:

(i) From A observe the opposite point B on the further bank, and pace off any distance AD at rt. angles to AB. From D observe the angle ADB, then by fixing a post at P mark a direction DPC, so that $A\hat{D}C = A\hat{D}B$ (Fig. 141).

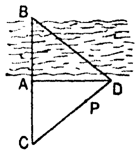


Fig. 141.

Finally, find a point C, which is in the same straight line with D, P, and also in a straight line with A, B. AC gives the breadth of the river.

(ii) Take a point B opposite to a point A on the further bank. Walk off in a direction BCD at rt. angles to AB, and mark off any distance BC, and take D so that CD = CB. Then walk off along DE at rt. angles to DB and take E, so that ECA is a straight line. DE gives the breadth of the river (Fig. 142).

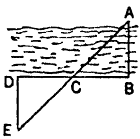


Fig. 142.

25. In a triangle ABC, EO, DO bisect AB, BC at right angles respectively. Prove that OA = OB = OC (Fig. 143).

26. Construct a rectangle (a quadrilateral with all the angles rt. angles and the opposite sides equal) ABCD such that AB = 3 ins., BC = 2 ins. Bisect the four angles by straight lines AP, BP meeting at P and CQ, DQ meeting at Q. Measure PQ. Then prove, without measurement, that AP, BP, CQ, DQ are all equal.

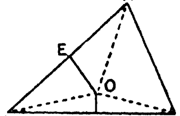


Fig. 143.

27. BAC is any angle. P and M are points on AB and Q and N are points on AC, such that AP = AQ and AM = AN.
If PN, QM cut at S, prove that AS bisects the angle BAC (Fig. 144).

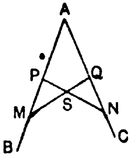


Fig. 144

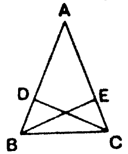


Fig. 145.

28. In a triangle ABC, the angles at B and C are equal, and CD, BE are perpendicular to AB, AC respectively. Prove that BD = CE and CD = BE (Fig 145)

29. ABCD is a parallelogram (*i e* a four-sided figure with its opposite sides parallel). Prove that AB = CD, BC = AD and $\hat{A} = \hat{C}$.
Note. Join BD.

30. ABC is a △, and a line parallel to BC meets AB in D, and AC in E. F is a point in AB such that BF = AD, and FG is drawn parallel to AC to meet BC at G. Prove that BG = DE.

SYMMETRY.

It is possible in plane geometry for a figure to be symmetrical about a line (or axis), or about a point.

(i) *Symmetry about a line.*

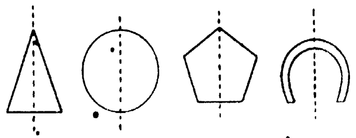


Fig. 146.

In the cases shown of an isosceles triangle, circle, regular pentagon and horse-shoe, there is symmetry about the dotted line, and if the figure is cut out and folded about the dotted line, then the right-hand half fits exactly on the left-hand half. Conversely, if a piece of paper is folded

and any figure cut out with the paper doubled, then on unfolding, a symmetrical figure is obtained, the crease being the axis of symmetry.

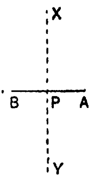


Fig. 147.

If a point A is taken on a piece of paper which is folded about XY, and the new position of A is B, then on joining AB (intersecting the crease in P) and folding up again, PA lies exactly on PB and $A\hat{P}X$ lies exactly on $B\hat{P}X$, from which we conclude that $PA = PB$ and $A\hat{P}X = B\hat{P}X = 90°$, since APB is a straight line.

A and B are symmetrically situated with respect to XY, and have the same relation as an object and its image when seen in a plane mirror.

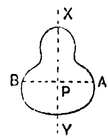


Fig. 148.

It follows that, in any figure symmetrical about XY, if we take any point A on the right-hand half and draw a perpendicular AP to XY, then AP produced meets the left-hand half in a point B, such that

$$AP = BP,$$
$$A\hat{P}X = B\hat{P}X = 90°.$$

If we find the image of a line AC in XY, we obtain BC, where as before $PA = PB$. Corresponding to any point R on AC, there will be a point Q on BC, where $RS = SQ$.

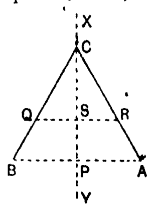


Fig. 149.

(ii) *Symmetry about a point.*

The best known case is that of a circle about its centre, but others are shown in the accompanying diagrams, each figure being symmetrical about O ; they may be considered

as symmetrical about an axis passing through O at right angles to the plane of the paper.

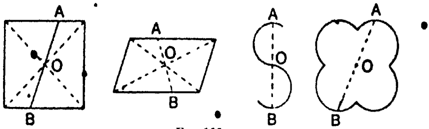


FIG. 150.

To any point, such as A, corresponds another point B, where OA = OB.

EXERCISE 15.

1. Indicate by a dotted line the lines of symmetry in (i) a kite,

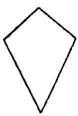


FIG. 151.

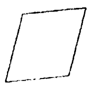


FIG. 152

FIG. 153.

FIG. 154.

(ii) a rhombus, (iii) segment of a circle, (iv) sector of a circle; also state if these figures are symmetrical about any point, and if so, which.

2. Indicate by rough diagrams, the lines of symmetry in a cart wheel, cylinder, cone, cube, six-pointed star, five-pointed star, envelope.

3. Draw block capital letters A, B, C, etc., and indicate the axes of symmetry by dotted lines.

4. Draw a line PQ on paper and through P draw any line XY. Fold the paper about XY as a crease, and let the new position (or image) of PQ be PR, so that PQ = PR. Unfold again and join QR (Fig. 155).

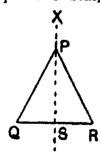


FIG. 155.

Notice that the $\triangle$PQR is isosceles.

Since XY is an axis of symmetry, what do you know about the angles PQR, PRQ? (See Theorem 3.)

5. Take any two points Q, R and bisect QR at right angles by XY. Then as we have already seen, XY is an axis of symmetry for Q and R. At Q and R draw *equal* angles and produce the arms to meet at P, then since the figure is symmetrical about XY, it follows that P, the point where the arms meet must be on XY. What is the connection between the lines PQ, PR? (See Theorem 4.)

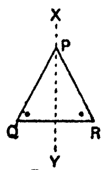


FIG. 156.

ISOSCELES TRIANGLES.

THEOREM 3.

If two sides of a triangle are equal, the opposite angles must also be equal. (See Ex. 15. 4.)

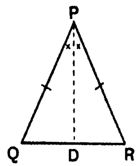


FIG. 157.

Let PQR be a $\triangle$ having $PQ = PR$.
Then we have to prove that

$$P\hat{Q}R = P\hat{R}Q.$$

Hypothetical Construction. Suppose PD is the line which bisects $Q\hat{P}R$.

Proof. In the $\triangle$s PQD and PRD,

(i) $PQ = PR$, (Hypoth.)

(ii) PD is common to both $\triangle$s,

(iii) the included $Q\hat{P}D$ = the included $R\hat{P}D$; (Constr.)

$$\therefore \triangle PQD \equiv \triangle PRD,$$

and in particular, $P\hat{Q}D = P\hat{R}D$.

Note (i) If the triangle PQR is equilateral, then $\hat{P} = \hat{Q} = \hat{R}$; but since the sum of these angles is 180°, it follows that each angle of an equilateral triangle = 60°.

(ii) PD bisects QR at right angles.

THEOREM 4.

If two angles of a triangle are equal, the sides opposite those angles must be equal. (See Ex. 15. 5.)

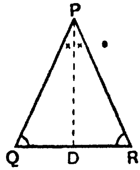


FIG. 158.

Let PQR be a $\triangle$ having $P\hat{Q}R = P\hat{R}Q$.

Then we have to prove that PQ = PR.

Hypothetical Construction. Suppose PD is the line which bisects $Q\hat{P}R$.

Proof. In the $\triangle$s PQD and PRD,

(i) $P\hat{Q}D = P\hat{R}D$, (Hypoth.)

(ii) $Q\hat{P}D = R\hat{P}D$, (Constr.)

(iii) PD is common to both $\triangle$s ;

$\therefore$ $\triangle PQD \equiv \triangle PRD$, (2 angles, 1 side)

and in particular, PQ = PR.

N.B.—This theorem is the **converse** of Theorem 3.

The results contained in Theorems 3 and 4 can be illustrated by two ordinary set squares, which are equal in all respects.

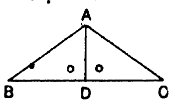


FIG. 159.

If they are placed with their short sides coincident, then BDC will be a straight line since the angles ADB and ADC are each 90°.

Thus ABC is an isosceles Δ, and AB = AC, $\hat{B} = \hat{C}$; moreover AD, which bisects angle BAC, bisects BC at rt. angles.

EXERCISE 16.

1. If each of the base angles of an isosceles triangle ABC is 55°, what is the size of the vertical angle ?

2. If the vertical angle of an isosceles △ is 54°, what is the size of each of the base angles ?

3. Given that the vertical angle of an isosceles △ is 142°, what is the size of each of the base angles ?

4. If the vertical angle of an isosceles △ is 90°, what is the size of each of the base angles ?

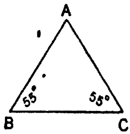


FIG. 160.

5. If one angle of an isosceles △ is 50°, find the other angles, showing that there are 2 solutions.

6. If the base BC of an isosceles △ is produced to D, and angle BAC = 32°, find the values of the angles ABC and ACD (Fig. 161).

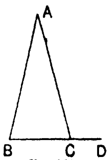


FIG. 161.

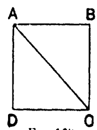


FIG. 162.

7. In a square ABCD, what angles does the diagonal AC make with the sides ? (Fig. 162).

8. In the Fig. of Qn 6, if angle ACD = 110°, find the values of the angles ABC and BAC

9. Take a line AB 4 cms. long. At A and B make $C\hat{A}B = C\hat{B}A = 50°$. Measure AC and BC.

10. From a point A in a line PQ draw a perpendicular AB to the line, making AB 5 cms. With B as centre, and radius 8 cms., draw a circle cutting PQ at H, K. Measure $B\hat{H}K$ and $B\hat{K}H$.

11. Draw XY 5·5 cms. long. With centres X and Y and radii each equal to 4·2 cms., draw arcs meeting in P. With centres X and Y, and radii each equal to 6·8 cms., draw arcs meeting on the same side of XY in Q. Join PX, PY, QX, QY.

Measure $Q\hat{X}P$, $Q\hat{Y}P$, PQ.

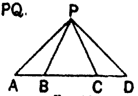


FIG. 163.

12. ABCD is a straight line, and P is a point outside it. Show that if PA = PD, and PB = PC, then AB = CD (Fig. 163).

13. The equal lines OK and OM bisect the angles $\hat{K}$ and $\hat{M}$ of a triangle HKM. Prove that the triangle must be isosceles.

14. XYZ is an isosceles triangle having XY = XZ. OY bisects $\hat{Y}$, and OZ bisects $\hat{Z}$. Prove that OY = OZ

15. A quadrilateral ABCD (Fig. 164) has AB = AD and CB = CD. Prove that $\hat{B} = \hat{D}$. (Such a figure is called a *kite*.)

16. ABCD is a quadrilateral having AB = AD and $\hat{B} = \hat{D}$. Prove that CB = CD

17. Prove that if ABC is an isosceles triangle with AB = AC, and D, E, F be the middle points of BC, CA, AB, then DF = DE.

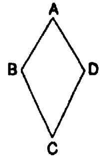


FIG. 164.

Thence if ABC is equilateral, prove that DEF is equilateral also.

18. In a triangle ABC, if O is the middle pt. of AC and OB = OA = OC, prove that $A\hat{B}C = 90°$ (Fig 165).

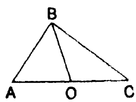


FIG. 165.

19. In a triangle ABC, the side BA is produced to D; show that, if the bisector of the angle CAD is parallel to BC, then the triangle ABC is isosceles.

20. ABC is a triangle having the sides AB and AC equal, and from any point D in AC a straight line DE is drawn parallel to AB, meeting BC in E; prove that DE and DC are equal.

21. ABC is an equilateral triangle. Points P, Q, R are taken in the sides AB, BC, CA respectively, so that AP = BQ = CR. Prove that PQR is an equilateral triangle.

22. If an isosceles △ABC has each of its base angles 72°, A being the vertex, and the angle C is bisected by a line meeting AB in D, prove that BC = CD = DA.

23. The internal and external bisectors of an angle BAC meet a line through C parallel to AB in P, Q. Prove that PC = CQ.

24. AB is a straight line 3 inches in length. Without using a protractor or set square, make an angle ABC = 120°, and measure off BC = 4 inches. Join AC and measure its length to the nearest tenth of an inch.

(Remember that the angle of an equilateral △ is 60°.)

25. If two sides AB, AC of a triangle ABC are equal, and BD, CE are drawn perpendicular to AC, AB and intersecting at O, prove that AO bisects the angle A.

26. In a regular pentagon ABCDE, if O is the centre of the ⊙ passing through the angular points, then remembering that all the sides of the pentagon are equal, what would you expect to be the size of each of the angles at O? Thus find the size of all the other angles of the 5 equal isosceles triangles

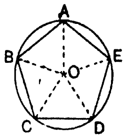


FIG. 166.

27. Work out similar problems for a regular
(i) hexagon, (ii) heptagon,
(iii) octagon

28. With centre A and radius AB a circle is drawn. With centre B and equal radius an arc is drawn intersecting the first circle at C. Similarly from centre C the point D is determined, and from centre D the point E. Prove that BAE is a straight line (Fig. 167)

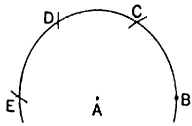


FIG. 167.

29. O is the centre of a circle passing through the angular points of the triangle ABC. If $\hat{AOB}$, $\hat{BOC}$, $\hat{COA}$ are respectively equal to 150°, 140°, 70°, what are the sizes of the angles OAB, OBA, OBC, OCB, OCA, OAC? (Fig 168).

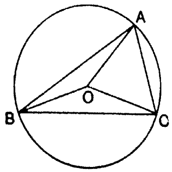


FIG. 168.

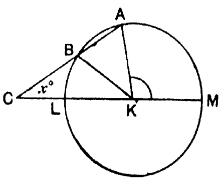


FIG. 169.

30. In Fig. 169 the chord AB of the circle is produced to C so that BC is equal to the radius, and CLKM is drawn through the centre K of the circle

Given that there is a simple relation between the angles MCA, MKA, find it, first taking MCA as 25°, then, generally, denoting MCA by $x°$. In each case sketch a large figure (not to scale) and write in each angle its value, showing with brief reasons how you determine the various angles.

Now apply the construction here suggested to trisect an angle MKA with the help of tracing paper. Draw a circle of 5 cms. radius, make the angle MKA 60°, and on the tracing paper draw a line CBZ of indefinite length, marking off CB equal to the radius 5 cms. State how you use the tracing paper in solving the problem and how you test your result. (*Army.*)

CONGRUENT TRIANGLES.

FACT 10.

Draw a line AB 6 cms. long. With A as centre and radius 3 cms., draw part of a circle as in Fig. 170. With B as centre and radius 5 cms., draw part of another circle. The two parts of the circle will meet at some point C; join AC and BC. You have now described a triangle with sides 3, 5 and 6 cms.

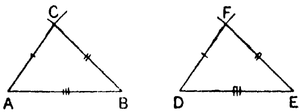


FIG. 170.

Try to draw another triangle DEF with these numbers, but of a different size and shape from ABC. Convince yourself as to the impossibility of this.

The Geometrical Fact is:

If the three sides of one triangle are respectively equal to the three sides of another triangle, the triangles must be congruent.

EXERCISE 17.

1. Draw two triangles as above, using the numbers, 3, 5, 7 cms. Measure all three angles Cut the triangles out and see that they can be made to coincide.

2. Repeat, using 2, 3, 4 inches

3. Take AB=6 cms, AC=2 cms, BC=4 cms, and notice that the triangle is a failure.

4. Take AB=6 cms., AB=2 cms, BC=3 cms. Again notice the failure.

Observe that *any two sides of a triangle must be together greater than the third side.*

5. Describe a triangle with sides 5, 5, 5 cms Measure the angles. Join one angular point to the middle point of the opposite side and measure this line.

6. Describe a triangle with sides 2·6, 3·8, 4·2 cms Bisect each side and join the points of bisection. Measure the sides of this smaller triangle.

7. A quadrilateral has its opposite sides equal. Prove that either diagonal divides the quadrilateral into two congruent triangles and that the opposite sides of the quadrilateral are parallel.

8. Prove that either diagonal of a quadrilateral with four equal sides (*i.e.* a rhombus or a square) bisects the angles through which it passes, and that the diagonals bisect one another at right angles.

9. Two isosceles triangles stand on the same base and on the same side of it. Prove that the line joining the vertices must bisect the vertical angles and, if produced, bisect the base.

10. Three straight lines (Fig. 171) AOP, BOQ, COR are drawn through O, so that OA=OP, OB=OQ, OC=OR. Prove that $\triangle ABC \equiv \triangle PQR$.

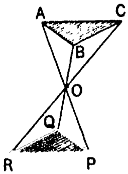


FIG. 171.

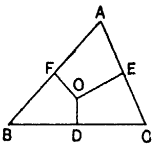


FIG. 172.

11. D, E, F are the middle points of the sides BC, CA, AB, of a triangle ABC. FO, EO are perp. to AB, AC respectively, and O, D are joined. Prove that OD is perp. to BC (Fig 172).
(Join OA, OB, OC.)

12. Two circles with centres at A and B intersect at X and Y. AX, AY, BX, BY are joined. Prove that AB bisects the angles XAY and XBY.
If XY cuts AB in Z, also prove that XZ=ZY and that XY and AB are at right angles (Fig. 173).

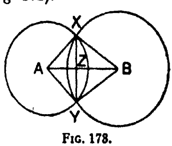


FIG. 173.

13. ABCD is a quadrilateral. The diagonals AC and DB intersect at K. AD=CB and AC=DB.

Prove (i) $\triangle ADB \equiv \triangle ACB$;
(ii) $\triangle KDA \equiv \triangle KCB$;
(iii) $\triangle CDA = \triangle DCB$;
(iv) $\triangle KAB$ and $\triangle KDC$ are isosceles.

14. PQRS is a quadrilateral in which PQ=RS. The lines bisecting PS and QR at right angles meet at O. Prove that $P\hat{O}Q = R\hat{O}S$.

15. ACB and ADB are two triangles on the same side of AB, such that AC=BD, AD=BC, and AD and BC intersect in P; prove that the triangle APB is isosceles.

DEF. The *hypotenuse* of a right-angled $\triangle$ is the side opposite the right angle.

THEOREM 5.

If the hypotenuse and one side of a right-angled triangle be respectively equal to the hypotenuse and one side of another right-angled triangle, the triangles must be congruent.

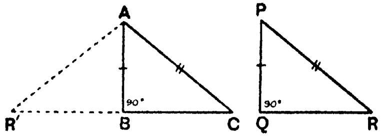


FIG. 174.

Let ABC and PQR be two right-angled $\triangle$s, having

$$\text{(i)}\quad \hat{B} = \hat{Q} = 90^\circ,$$
$$\text{(ii)}\quad AC = PR,$$
$$\text{(iii)}\quad AB = PQ.$$

Then we have to prove that

$$\triangle ABC \equiv \triangle PQR.$$

Proof. Let $\triangle PQR$ be applied to $\triangle ABC$, so that

(i) PQ coincides with the equal line AB,

and (ii) R falls on the side of AB opposite to C, at R′.

$\triangle AR'B \equiv \triangle PRQ$ because it is the same triangle in a different position.

Since $\quad A\hat{B}C + A\hat{B}R' = 90^\circ + 90^\circ = 180^\circ,\quad$ (Hypoth.)

$\therefore$ R′BC is a straight line.

Also $\quad AC = PR = AR',$

$\therefore\ A\hat{R}'C = A\hat{C}R',\quad$ ($\triangle ACR'$ isosceles)

i.e. $P\hat{R}Q = A\hat{C}B.$

Hence the two $\triangle$s ABC and PQR have

$$\text{(i)}\quad \hat{B} = \hat{Q},$$
$$\text{(ii)}\quad \hat{C} = \hat{R},$$
$$\text{(iii)}\quad AB = PQ;$$
$$\therefore\ \triangle ABC \equiv \triangle PQR.$$

EXERCISE 18.

1. Draw a right angle and take a point 5 cms. from the angle on one arm. With this point as centre draw a circle, radius 13 cms. cutting the other arm of the right angle. Join the two points on the arms.

You have described a right-angled triangle with hypotenuse 13 cms. and another side 5 cms. Measure the third side.

Can you describe a different shaped right-angled triangle with hypotenuse 13 and another side 5 cms.? Give reasons for your opinion.

2. Draw a circle radius 4 cms. and AB any diameter. With centre B and radius 3 cms. draw another circle, cutting the first at H, K. Join HA and KA, also HB and KB. Measure the lines HA, KA. Also the angles H, K.

Can you give a reason why HA = KA? This is a method for constructing a right-angled triangle, given the hypotenuse and another side.

3. Repeat Qn. 1, using numbers 4·3 and 7 cms.

4. From a point K between the lines OA and OB, perpendiculars are drawn to OA and OB; if these perpendiculars are equal, prove that OK must bisect the angle AOB.

5. S is the centre of a circle, AB any chord, SK is drawn perpendicular to AB. Prove that AK = BK. [Join SA, SB, and remember these lines are equal, because they are radii of the same circle.]

6. ABC is a triangle, BH is drawn perpendicular to AC, and CK perpendicular to AB; if BH = CK prove that ABC must be isosceles.

7. S is the centre of a circle, AB and CD are two equal chords to which SK and SM are perpendicular. By Qn. 5, K and M are the middle points of the chords. Prove SK = SM.

8. If D is the middle point of the base BC of a triangle ABC, and DE, DF are drawn perp. to AB, AC respectively, prove that the triangle is isosceles if DE = DF.

9. OB and OC are the bisectors of the angles B and C of a triangle ABC. Prove that OA must bisect the angle A (Fig. 175).

(Draw perps. from O to the three sides.)

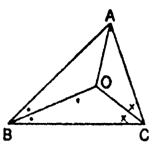


FIG. 175

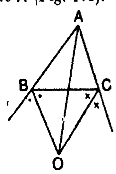


FIG. 176.

10. ABC is a triangle having AB and AC produced; OB and OC are the bisectors of the exterior angles at B and C. Prove that OA must bisect the angle A (Fig. 176).

(Draw perps. from O to BC and also to AB, AC produced.)

AMBIGUOUS CASE.

We have already seen in Fact 8 that only one △ can be constructed if we are given the magnitudes of two sides and the *included* angle ; it is now necessary to investigate the case in which we are given the magnitudes of two sides and an angle *opposite* one of those sides.

(i) Suppose we are given that $\hat{C} = 37°$, $AC = 6{\cdot}5$ cms., $AB = 4{\cdot}4$ cms., and wish to construct the triangle.

Draw $CA = 6{\cdot}5$ cms. and make $A\hat{C}X = 37°$; with A for centre and 4·4 cms. for radius construct an arc of a circle to cut CX in *two* points B_1 and B_2 ; join AB_1, AB_2.

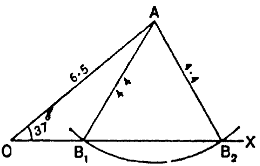


FIG 177

It will be seen that each of the △s ACB_1 and ACB_2 satisfies the given data, *i.e.* there are *two* triangles.

(ii) If $\hat{C} = 37°$, $AC = 4{\cdot}4$ cms., $AB = 6{\cdot}5$ cms., then draw $AC = 4{\cdot}4$ cms. and make $A\hat{C}X = 37°$.

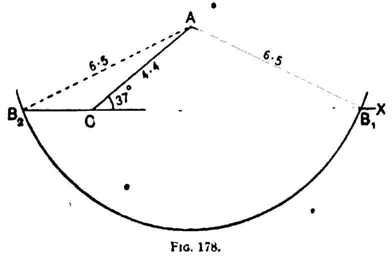


FIG. 178.

With A for centre and 6·5 cms. radius, construct an arc of a circle to cut CX, or CX produced backwards, at B_1 and B_2 ; join AB_1, AB_2.

In the $\triangle ACB_2$, though the two sides AB_2, AC have the correct magnitudes, it should be noticed that the angle C, *i.e.* ACB_2 is not 37°, but the supplement of 37°.

The $\triangle ACB_1$ satisfies the given data.

Thus, in this case, there is only *one* triangle.

(iii) With given values of C, AC, AB, it may happen that the arc of the circle touches the line CX at a point B (say).

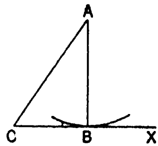


FIG. 179.

In this case there is only *one* triangle, and we shall prove later that $A\hat{B}C = 90°$.

(iv) It may happen that the arc of the circle does not meet the line CX, and then it will be impossible to construct even one triangle.

We thus see that *two solutions* will not arise unless *the side opposite the given angle is the smaller of the two given sides.*

The result of (i) and (ii) might be stated as the following Theorem:

If two sides of one $\triangle$ are equal to two sides of another, and a non-included angle in one equals the corresponding angle in the other, the triangles are either congruent or contain a pair of supplementary angles.

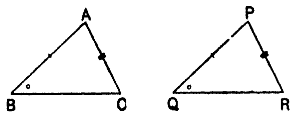


FIG. 180.

In Fig. 180, angles C and R are equal.

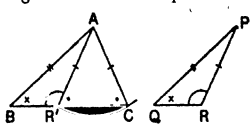


FIG. 181.

In Fig. 181, angles C and R are supplementary.

SIMILAR TRIANGLES.

If two triangles have their three angles respectively equal then they will be of the same shape but different sizes, *i.e.* they will not necessarily be congruent.

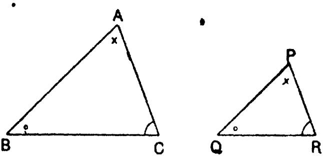


FIG. 182.

Suppose we construct a triangle ABC with BC = 5 cms., $\hat{B} = 60°$, $\hat{C} = 65°$, and consequently $\hat{A} = 55°$; it is possible to copy this on any scale, so that the copy PQR has QR = 3 cms., $\hat{Q} = 60°$, $\hat{R} = 65°$; *i.e.* the two triangles have their angles respectively equal, but they are not congruent.

Such triangles are said to be *similar*, PQR and ABC are drawn to scale, and the corresponding sides have the same ratio to one another, *i.e.*

$$\frac{QR}{BC} = \frac{QP}{BA} = \frac{RP}{CA}.$$

In the case of *polygons*, it will be shown later that they may be equiangular without having the corresponding sides in the same ratio.

The Diagonal Scale.

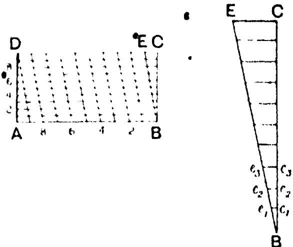
With an ordinary ruler it is possible to measure the length of a line to a tenth of an inch, and to guess at the second decimal place; but with a Diagonal Scale the nearest hundredth may be obtained exactly.

Take AB 1 inch and divide it into tenths, and ABCD a rectangle with AD of some suitable length; divide AD and DC each into 10 equal parts.

Through the divisions on AD draw lines parallel to AB, and through the divisions on DC draw lines to the graduations on AB in the *diagonal* way shown in Fig. 183.

Make an enlarged diagram of ECB where EC represents $\frac{1}{10}$ inch.

Since e_1c_1 is $\parallel$ to EC, $\therefore$ the $\triangle$s Be_1c_1 and BEC are equiangular and similar,



$$\therefore \frac{e_1c_1}{Bc_1} = \frac{EC}{BC},$$

$$i.e. \quad \frac{e_1c_1}{EC} = \frac{Bc_1}{BC} = \frac{1}{10};$$

Fig. 183.

$$\therefore \; e_1c_1 = \tfrac{1}{10}EC = \tfrac{1}{100} \text{ inch.}$$

Similarly $$\frac{e_2c_2}{EC} = \frac{Bc_2}{BC} = \frac{2}{10};$$

$$\therefore \; e_2c_2 = \tfrac{2}{10}EC = \tfrac{2}{100} \text{ inch};$$

$$e_3c_3 = \tfrac{3}{100} \text{ inch, etc.}$$

If we wish to draw a line 2·57 inches long, place one point of a pair of dividers at G (the intersection of the diagonal line through 5 tenths of an inch on OX and the horizontal line through the 7 division on the line through X at right angles to XO), and the other point at F on the same horizontal line as G, and on the vertical line through 2 inches.

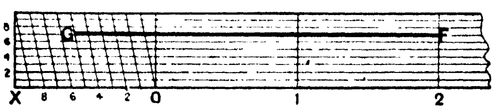


Fig. 184.

It should be noticed that the right-hand large numbers give the *inches*, the left-hand small numbers the *tenths*, and the numbers on the vertical line through X the ***hundredths*** of inches.

EXERCISE 19.

1. Draw two triangles with AB = 7 cms., BC = 3·8 cms. and $\hat{A} = 30°$. Measure AC in each triangle. Discuss the case when AB = 7 cms., BC = 3·5 cms., $\hat{A} = 30°$; and also when AB = 7 cms., BC = 3·2 cms., $\hat{A} = 30°$.

2. Draw two triangles with BC = 4 cms., AC = 3·5 cms., $B = 40°$. Measure AB in each triangle. Discuss the cases when AC = 2·6 cms. and 2·4 cms.

3. Draw two triangles with $a = 8$ cms., $b = 7$ cms., $\hat{B} = 40°$. Measure c in each triangle. Discuss the cases when $b = 5·14$ cms. and 4·8 cms.

4. Draw two triangles taking $c = 6$ cms., $a = 3$ cms., $\hat{A} = 24°$. Find a value for a which would make it just impossible to draw two triangles.

5. Draw two triangles, taking $a = 7$ cms., $b = 5$ cms., $B = 34°$. Measure c in each triangle. Discuss the case when $B = 45\frac{1}{2}°$.

6. Fig. 185 shows the essential parts of a simple piece of mechanism, AB and AC being jointed rods, AC pivoted to DE at C, and B capable of moving to and fro in a slot along DE (supposed of indefinite length). Take AB as 2·5 cms. and AC as 4·5 cms., indicate on a drawing the range of movement possible for B and for A, and also the total angle through which the rod AC is free to turn.

If $A\hat{C}B = 20°$, what are the two possible values for CB? (*Army*.)

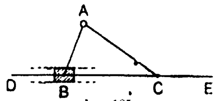


FIG. 185

7. Draw a △ABC with $a = 6$ cms., $b = 4$ cms., $c = 5$ cms. Draw BB′ between BA and BC so that angle CBB′ = 25° and BB′ = 7 cms. Draw B′C′ parallel to BC and equal to 3 cms. Construct a △A′B′C′ equiangular to △ABC. Join AA′, BB′, CC′ and produce. They should meet in a point X. Measure AX, A′X, A′B′, A′C′.

8. Draw a quadrilateral ABCD with AB = 5 cms., AD = 4·5 cms., BC = 3·6 cms., $B\hat{A}D = 70°$, $A\hat{B}C = 60°$. Join AC. Take B′ such that AB′ = 2·5 cms.

Draw B′C′ ∥ BC, C′D′ ∥ CD.

Find the values of $\dfrac{B'C'}{BC}, \dfrac{C'D'}{CD}, \dfrac{AD'}{AD}$.

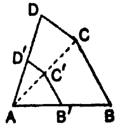


FIG. 186.

9. A quadrilateral ABCD has the adjacent sides AB, AD equal, and also the opposite angles B, D equal. Prove that its diagonals are perpendicular to each other.

CONSTRUCTION 3.

To draw a perpendicular to a given straight line from a given point in the line.

Let AB be the given line and C the given point in it.

Construction. With centre C and any radius $<$ AC and $<$BC, describe a ⊙ cutting AB in X, Y.

With X and Y for centres, and any radius $>$XC, describe two ⊙s cutting at D.

Join DX, DY, DC.

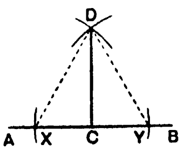


Fig. 189.

Proof. In the △s DCX, DCY,

DX = DY,	(equal radii)
CX = CY,	(radii)
DC is common,	
∴ △DCX ≡ △DCY,	(3 sides)

and in particular $\widehat{DCX} = \widehat{DCY}$;

but these are adjacent angles, and thus right angles,

∴ DC is perp. to AB.

CONSTRUCTION 4.

To draw a perpendicular to a given line from a given point outside the line.

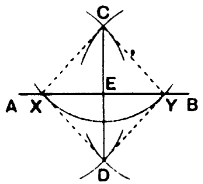


FIG. 100.

Let AB be the given line and C a point outside it.

Construction. With centre C, describe a ⊙ cutting AB in X, Y.

With centres X and Y, and radius XC, draw two circles cutting in C, D.

Join CX, CY, DX, DY, CD cutting AB at E.

Proof. In the △s CXD, CYD,

CX = CY, (radii)

XD = YD, (equal radii)

CD is common,

∴ △CXD ≡ △CYD, (3 sides)

and in particular, $X\hat{C}D = Y\hat{C}D$.

In the △s CEX, CEY,

CX = CY, (radii)

CE is common.

$X\hat{C}E = Y\hat{C}E$, (proved above)

∴ △CEX ≡ △CEY, (2 sides, included angle)

and in particular, $C\hat{E}X = C\hat{E}Y$;

but these are adjacent angles, and thus right angles,

∴ CE is perp. to AB.

(ii) Proceeding to the two points A and C, the angles ACD, CAD are now observed, and it is consequently possible to represent the $\triangle$ACD.

The original measured distance AB is called the *base*, and the points fixed by measuring the angles are known as the intersected stations or *intersections*.

If we have a choice, it is advisable to select the points C, D, etc., so that the angles BCA, CDA, etc., are fairly large, and, as a matter of practice, about 60° or 120° is found to be suitable. If the angles are small, it is difficult to plot the points C, D, etc., accurately, and with a large number of points, an angle of 90° often causes a good deal of coincidence of various lines or *rays* and thus produces confusion in the figure.

The usual method of finding the distance AB is by pacing, and with a little practice this can be done sufficiently accurately for practical purposes.

As a matter of practice, the two angles BAC and BAD would probably be measured at A and the difference of these angles would give the angle CAD required for (ii), the angles being measured by means of a prismatic compass.

EXERCISE 20.

In these examples nothing but ruler and compasses are to be used when the word **construct** *occurs. All lines used in the constructions must be shown by* thin *or* dotted *lines, and the constructions must be described in words.*

1. Draw a triangle $a=6$; B$=64°$, A$=78°$. Construct a line bisecting BC at right angles and meeting AC at some point K Measure KB and KC.

2. Draw a triangle $a=7$; $b=8$; $c=5$. Construct the bisector of A, meeting BC at some point K. Measure KB and KC, and compare the ratio of these with the ratio $c:b$.

3. Draw a triangle $c=5$; $b=8$; A$=48°$ Construct a line through H the middle point of AB, parallel to BC; let it meet AC at some point K. Measure HK, KA, KC.

4. Draw a triangle $a=4$; $b=5$; $c=7$. Construct lines bisecting BC and CA at right angles, and let them meet at O. Measure OA, OB, OC.

5. Draw a triangle $b=4$; $c=6$; A$=52°$. Construct a perpendicular from C to AB and another from B to CA. Measure these

perpendiculars. If they meet at Q, join AO, and measure with your protractor the angle between AO produced and BC.

6. Draw a triangle $a=7$; B $=68°$; C $=50°$. Bisect these angles with your protractor, and let the bisectors meet at I. Construct perpendiculars from I to the three sides, and measure these perpendiculars.

7. Draw a triangle $a=6$, $b=8$, C $=55°$. Construct the bisectors of the angles A and B, and let these bisectors meet at O. Join OC, and measure the two parts into which this line divides $\hat{C}$.

8. Draw a triangle $a=7$, $b=6$, $c=9$. Take points H and K in AB, so that AH = HK = 3.
Through H and K construct parallels to BC, meeting AC at M and N. Measure AM, MN and NC.

9. Draw a straight line AB, 2·5 inches long; also a straight line BC, 1·4 inches long, perpendicular to AB. Measure the angle BAC.

10. Draw a straight line AB, 2·6 inches long, and a line AC making with AB an angle of 30°. Draw also through B a line BC at right angles to AB, and measure AC and BC.

11. On a base of 2 inches draw a triangle whose base angles are 40° and 100°. Construct the perpendicular from the vertex to the base, and measure its length.

12. Draw a triangle ABC having BC = 3 ins., $A\hat{B}C=32°$, and $A\hat{C}B=105°$. Construct the bisector of the angle BAC, meeting BC in D. Measure BD.

13. Construct a triangle whose sides are 2·4 ins., 3·2 ins., and 4·9 ins. Measure the smallest angle and bisect the largest, measuring the length of this bisector as far as the opposite side.

14. D is a point in the side AB of a triangle ABC such that DB=DC=3 inches, the angle ABC=53°, and the angle DCA=47°. Draw the triangle, and measure the length of AD.

15. Draw a parallelogram with sides of lengths 7·8 cms. and 5·4 cms., and having an angle of 70°. Measure and compare the lengths of the diagonals.

16. Draw a quadrilateral ABCD having AB=3; BC=7; CD=6; B=42°; and C=68°. Measure AD.

17. Draw a quadrilateral ABCD having AB=4; BC=6; AC=7; AD=5; CD=8. Measure B and D.

18. Draw a quadrilateral ABCD having AB=4; BC=5; AD=7; CD=8; and B=64°. Measure BD.

19. Draw a quadrilateral ABCD having $\hat{B}=60°$; $\hat{C}=100°$; $C\hat{A}D=36°$; $A\hat{C}D=72°$; BC=7. Measure BD.

20. Draw a quadrilateral ABCD having $\hat{B}=40°$; $\hat{C}=96°$; $\hat{D}=80°$; $C\hat{A}D=30°$; AB = 5. Measure BD.

21. Draw a quadrilateral in which two adjacent sides are each 1·6 ins., the other sides are 1·1 ins and 2·3 ins., and the angle between them 105°. Measure the lengths of the diagonals.

22. Draw the plan of a piece of ground ABCD given that

(i) AB = 1000 yds., $A\hat{B}D=36°$, $B\hat{A}D=87°$, $D\hat{B}C=44°$, $B\hat{D}C=70°$;

(ii) AB = 1200 yds, $A\hat{B}D=35°$, $B\hat{A}D=80°$, $D\hat{B}C=57°$, $B\hat{D}C=62°$;

(iii) AB = 880 yds., $A\hat{B}D=42°$, $B\hat{A}D=64°$, $D\hat{B}C=32°$, $B\hat{D}C=27°$;

(iv) AB = 750 yds, $A\hat{B}D=55°$, $B\hat{A}D=48°$, $D\hat{B}C=28°$, $B\hat{D}C=25°$.

In each case what are the distances BC, DC?

23. Draw the quadrilateral ABCD, such that AB = 2 inches, BC = 1·5 inches, CD = 2·25 inches, the angle ABC = 120°, and the diagonals AC and BD are at right angles.
Measure the length of AD.

24. The figure represents a folding deck-chair, the prop FD

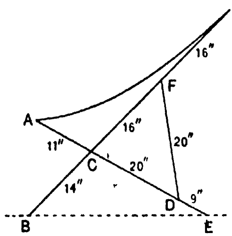


FIG. 105.

fitting into a notch at D. Determine by drawing the height of A above the ground.

When the chair is folded up with CA along CB, the canvas AG is taut; how much longer is it in the position in the figure than the straight line AG? (*Army*.)

THEOREM 6.

If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

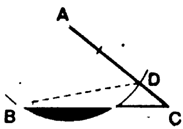


FIG. 196.

Let ABC be a triangle having AC > AB.

We have to prove that $\hat{ABC} > \hat{ACB}$.

Construction. From AC cut off AD = AB.

Join BD.

Proof. $\because$ AB = AD,

$\therefore \hat{ABD} = \hat{ADB}$. (Isos. Δ)

But $\hat{ADB}$ (the ext. angle of ΔBDC) > $\hat{BCD}$ (int. opp. angle),

$\therefore \hat{ABD} > \hat{BCD}$,

i.e. $\hat{ABD} > \hat{ACB}$;

but $\hat{ABC} >$ its part $\hat{ABD}$,

$\therefore \hat{ABC} > \hat{ACB}$.

THEOREM 7.

If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

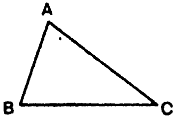


FIG. 107.

Let ABC be a triangle having $\hat{B} > \hat{C}$.

We have to prove that AC>AB.

Proof. There are three and only three possible suppositions concerning the relative lengths of AB and AC ;

(i) AB might be equal to AC,
(ii) AB ,, greater than AC,
(iii) AB ,, less than AC.

Now (i) would necessitate

$$\hat{B} = \hat{C}, \qquad \text{(Isos. } \Delta.\text{)}$$

and this is impossible by Hypothesis ;

(ii) would necessitate

$$\hat{C} > \hat{B}, \qquad \text{(Th. 6.)}$$

and this is also impossible by Hypothesis ;

∴ (iii) must be correct,

i.e. AB<AC or AC>AB.

COR. *The hypotenuse is the greatest side of a right-angled triangle.*

THEOREM 8.

Of all straight lines which can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.

PQ is the given straight line and A a point outside it.

AK is drawn perp. to PQ.

It is required to prove that AK is shorter than any other line drawn from A to PQ.

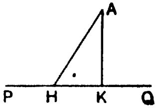


FIG. 198.

Construction. Any other point H is taken in PQ and AH joined.

Proof. Since the 3 angles of a triangle make up 180°,

and $\widehat{AKM} = 90°$,

$$\therefore \widehat{AHK} < 90°, \text{ i.e.} < \widehat{AKH}.$$

In the ΔAHK, because $\widehat{AHK} < \widehat{AKH}$,

and the smaller angle has the smaller side opposite to it,

$$\therefore AK < AH.$$

Since AH was any other line from A to PQ, it follows that AK is the shortest of all these lines.

Note. Compare this result with Theorem 1, Cor. 4 (page 51).

We are all familiar with the fact that the shortest distance between two points is a straight line, and thus if we wish to go from A to B, it will be shorter to go direct than travel through the point C; hence

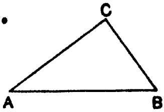


FIG. 199.

two sides of a Δ must be together greater than the third.

A formal proof of this result is given in Ex. 21. 6, 14.

EXERCISE 21.

1. Describe a triangle with sides 3, 5, 6 cms., and measure the angles opposite the sides 3 and 6.

2. Repeat Qn. 1, using numbers 3, 8, 9.

3. Describe a $\triangle$ with angles 40°, 60°, 80°, the side opposite 60° being 6 cms. Measure the other two sides.

4. In a $\triangle ABC$, arrange the sides a, b, c in order of magnitude :
(i) $A=60°$, $B=45°$, (ii) $B=72°$, $C=53°$;
(iii) $C=54°$, $A=37°$; (iv) $A=22°$, $B=84°$.

5. P is any point in a triangle ABC ; $AB=AC$ and $PB>PC$. Prove that $P\hat{B}A>P\hat{C}A$.

6. AK bisects the angle BAC of a triangle, and meets the side BC at K. Prove that $AB>BK$; $AC>CK$. Hence notice any two sides of a triangle are greater than the third

(Note that $A\hat{K}C$ is ext. angle of $\triangle ABK$, and $A\hat{K}B$ is ext. angle of $\triangle AKC$.)

7. ABC is an isosceles $\triangle$ with $AB=AC$; X and Y are any points in BC and BC produced ; prove that

$$AB>AX,$$
$$AB<AY.$$

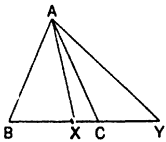


FIG. 200

8. In any $\triangle ABC$, prove that $AB\sim AC<BC$, *i.e.* the difference between *any* two sides is less than the third side.

9. If D is any point in the base BC of a $\triangle ABC$, prove that

$$AB+BC+CA>2AD.$$

10. If O is any point within a $\triangle ABC$, prove that

$$AB+BC+CA<2(OA+OB+OC).$$

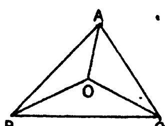


FIG. 201.

11. In any quadrilateral, why is the sum of three sides together greater than the fourth side ?

12. In a quad. ABCD, prove that the perimeter is

(i) $>$ the sum of the diagonals AC, BD,

(ii) $<$ twice the sum of the diagonals (consider the Δs OAB, OBC, OCD, ODA),

(iii) $>$ twice either diagonal (Fig. 202).

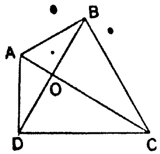


FIG. 202.

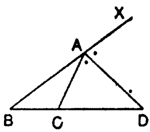


FIG. 203.

13. If the bisector of the exterior angle XAC of the ΔABC cuts BC produced at D, then AB$>$AC (Fig. 203)

(Note the connection between $X\hat{A}D$ and $A\hat{B}C$, and also between $D\hat{A}C$ and $A\hat{C}B$.)

14. In a ΔABC (Fig 204), by producing BA to K, so that AK$=$AC, show

(i) $B\hat{C}K > B\hat{K}C$; (ii) BA$+$AC$>$BC.

(Note that BA$+$AC$=$BK.)

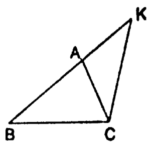


FIG. 204.

15. ABCD is a quad. in which the side AB is equal to the diagonal AC; prove that the side CD$<$diagonal BD.

16. If D is any point inside the triangle ABC (Fig. 205), prove that

(i) $B\hat{D}C > B\hat{A}C$; (ii) BA$+$AC$>$BD$+$DC.

(Produce BD to K; show that BA$+$AC$>$BK$+$KC$>$BD$+$DC.)

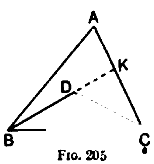


FIG. 205

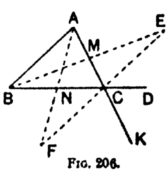


FIG. 206.

17. If one side BC of a ΔABC (Fig. 206) is produced to D, AC bisected at M, BM produced to E, so that ME$=$BM, prove that

(i) ΔAMB$\equiv\Delta$CME, (ii) $A\hat{C}D > B\hat{A}C$,

(iii) if BC is bisected at N, NF made equal to AN, and AC produced to K, $B\hat{C}K$ (*i.e.* $A\hat{C}D$)$>A\hat{B}C$,

i.e. the ext. angle of a Δ is greater than either of the interior opposite angles.

18. Assuming that an image I of an object O held before a mirror MR appears as far behind the mirror as the object itself is in front, IO being perpendicular to MR, compare the length IS from image to spectator's eye S with the length OP + PS, the path of the ray of light from object to spectator as reflected by the mirror, and justify your statement; also show that $O\hat{P}M = S\hat{P}R$.

Now imagine any other point K taken in MR on either side of P, and show that OK + KS is greater than OP + PS.

(Prove $\triangle$OMP $\equiv$ $\triangle$IMP; join IK and note OK = IK.)

Also prove that IS > OS

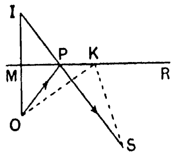


Fig. 207

19. If two mirrors AB, AC are at right angles and O is some object; let a ray of light pass to the spectator's eye along the path ODES. Let OM be perp. to AC.

If AD = 3·5 cms., AE = 5·2 cms., DM = 1·5 cms., draw the figure to scale, and mark the position of the image I in the mirror AC, and thence the image I′ of this in the mirror AB. Measure I′S, if ES = 4 cms.

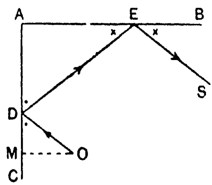


Fig. 208.

20. If the two sides PQ, PR of a $\triangle$PQR are equal respectively to the two sides AB, AC of a $\triangle$ABC, but the included angle QPR > the included angle BAC, prove that

base QR > base BC.

21. If two sides AB, AC of a $\triangle$ABC are equal respectively to two sides PQ, PR of a $\triangle$PQR, and the base BC > base QR, prove that

vertical angle BAC > vertical angle QPR.

DEF. A **parallelogram** is a four-sided figure with its opposite sides parallel.

THEOREM 9.

The opposite angles of a parallelogram are equal, and *conversely*, **if a quadrilateral has its opposite angles equal, then it must be a parallelogram.**

(i) Let ABCD be a ▱^{m}.

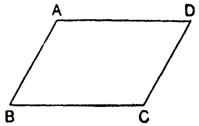


FIG. 209

We have to prove that

$$\hat{A} = \hat{C}, \text{ and } \hat{B} = \hat{D}.$$

Proof. ∵ AD ∥ BC,

$$\therefore \hat{A} + \hat{B} = 180^\circ. \qquad \text{(int. angles)}$$

∵ AB ∥ DC,

$$\therefore \hat{A} + \hat{D} = 180^\circ. \qquad \text{(int. angles)}$$

$$\therefore \hat{B} = \hat{D}.$$

Similarly, $\hat{A} = \hat{C}$.

(ii) Let ABCD be a quad. in which $\hat{A} = \hat{C}$ and $\hat{B} = \hat{D}$. We have to prove that ABCD is a parallelogram.

Proof. Since the sum of the 4 angles of any quad. is 360°,

$$\therefore \hat{A} + \hat{B} + \hat{C} + \hat{D} = 360^\circ,$$

$$i.e.\ 2(\hat{A} + \hat{B}) = 360^\circ,$$

$$\therefore \hat{A} + \hat{B} = 180^\circ, \qquad \text{(int. angles)}$$

$$\therefore \text{AD} \parallel \text{BC};$$

Similarly, AB ∥ DC;

$$\therefore \text{ABCD is a ▱}^m. \qquad \text{(definition)}$$

THEOREM 11.

The diagonals of a parallelogram bisect each other ; *conversely,* **if the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram.**

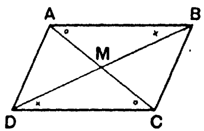


FIG. 213.

(i) Let ABCD be a parallelogram with its diagonals meeting at M.

We have to prove that

$$AM = MC,$$
$$DM = MB.$$

Proof. In the $\triangle$s AMB and CMD,

	$A\hat{B}M = C\hat{D}M,$	(alt. angles)
	$B\hat{A}M = M\hat{C}D,$	(alt. angles)
	$AB = DC,$	(opp. sides of ▭^{m}.)
	$\therefore \triangle AMB \equiv \triangle CMD,$	(2 angles and 1 side)
and in particular,	$AM = MC,$	(opp. equal angles)
	$DM = MB.$	

(ii) Let ABCD be a quad. in which AM = MC, BM = MD. We have to prove that ABCD is a ▭^{m}.

Proof. In the $\triangle$s AMB and CMD,

	$AM = MC,$	(hypoth.)
	$BM = MD,$	(hypoth.)
	$A\hat{M}B = C\hat{M}D,$	(vert. opp.)
	$\therefore \triangle AMB \equiv \triangle CMD,$	(2 sides and incl. angle)
and in particular,	$A\hat{B}M = C\hat{D}M,$	
	$\therefore AB \parallel DC.$	(alt. angles)

Similarly it can be proved that AD ∥ BC.

$\therefore$ ABCD is a ▱^{m}.

COR. In the case of a rhombus, in the Δs MAD and MAB,

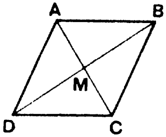


FIG 214

we have AD = AB,

DM = BM,

AM is common ;

$\therefore$ $\triangle$MAD = $\triangle$MAB ; (3 sides)

$\therefore$ $\hat{AMD} = \hat{AMB} = 1$ rt. angle.

Thus the diagonals of a rhombus *bisect one another at right angles.*

Also, $\hat{ADM} = \hat{ABM}$;

but $\hat{ABM} = \hat{MDC}$, (alt. angles)

$\therefore$ $\hat{ADM} = \hat{MDC}$,

i.e. the diagonal BD bisects the angle D.

Similarly all the *angles of the rhombus are bisected by the diagonals.*

THEOREM 12.

If one pair of opposite sides of a quadrilateral are equal and also parallel, the quadrilateral must be a parallelogram.

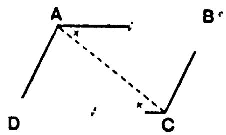


FIG. 215.

Let ABCD be a quadrilateral having

(i) AB || DC,

and (ii) AB = DC.

Then we have to prove that ABCD must be a ▭^{m}.

Construction. Join AC.

Proof. Because AB || DC ;

$\therefore$ $B\hat{A}C = A\hat{C}D$. (alt. angles)

Then in the two $\triangle$s ABC and CDA,

AB = DC, (hypoth.)

AC is common,

$B\hat{A}C = D\hat{C}A$;

$\therefore$ $\triangle ABC = \triangle CDA$, (2 sides and incl. angle)

and in particular,

$A\hat{C}B = D\hat{A}C$, being angles opposite equal marked sides ;

$\therefore$ AD || BC ; (alt. angles)

$\therefore$ ABCD is a ▭^{m}.

N.B.—This Theorem may be enunciated thus: ***If two straight lines are both equal and parallel, the joins of their extremities towards the same parts must be equal and parallel.***

EXERCISE 22.

1. Construct a ▱^{m} ABCD such that AB=$3\frac{1}{4}$ ins., AD=$2\frac{1}{4}$ ins., and the angle BAD=45°. Let the diagonals cut at O. Measure AO, CO, BO, DO.

2. Draw ABC and BCD right angles on opposite sides of BC, making AB=7 cms., BC=6 cms., CD=7 cms.
Measure AC and DB, and explain the connection between these two lengths.
Measure AD, and calling the point of intersection of AD and BC M, measure MC, MB, MA, MD.

3. Draw two lines at an angle of 30° with one another. Place between these lines another line 1 inch in length and terminated by them, so as to be perpendicular to one of them. Measure the lengths of the other sides of the right-angled triangle so formed.

4. Construct a quadrilateral ABCD, in which AB and DC are parallel, such that AB=4·5 inches and BC=CD=DA=2 inches. What is the length of each diagonal ?

5. Draw a rectangle whose diagonals are 4·2 inches in length and two of whose sides are 3·7 inches in length.
Measure each of the other two sides (Fig 216).
(Construct △ ABO)

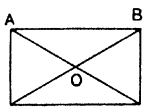


Fig. 216.

6. Draw a ▱^{m} with its diagonals 4·4 and 6·4 cms. in length, and one pair of equal sides 3·5 cms. Measure the angles and the other sides.

7. Draw a ▱^{m} with diagonals 4·2 and 5·4 cms , and the angle between them 60°. Measure the sides.

8. Construct a trapezium with the parallel sides 5·2, 9·2 cms., and the other two sides 3 and 6 cms. respectively. Measure the diagonals.

9. Construct a trapezium with the two parallel sides 4·2, 7·6 cms., and the other two sides 2·4, 4·3 cms respectively.
Bisect these last two sides at A and B, and measure AB.

10. From the opposite sides AT, SB of a parallelogram ABST two equal lengths AX, SY are cut off, and AY, SX are joined ; prove that AY=SX.

11. On opposite sides of a straight line AB are described two parallelograms ABQP and ABSR. Prove that PRSQ is a parallelogram.

12. ABC is any triangle; the bisector of the angle ACB meets AB in E; EF drawn parallel to BC meets AC in F; ED drawn parallel to AC meets BC in D. Prove that CFED has its four sides equal.

13. Make an angle AOB $=50°$. In OA take OP $=2\cdot5$ inches, and in OB take OQ $=1\cdot5$ ins. Draw PX parallel to OB and QX parallel to OA, intersecting at X. Show how to draw through X a line MN, cutting OA in M and OB in N, so that MN is bisected at X. Give reasons Measure PQ, MX, XN.

14. The side AB of a triangle ABC is bisected in E, and the parallelogram EBCH is completed. Prove that AECH is a parallelogram, and deduce that EH bisects AC

15. Any straight line drawn through the intersection of the diagonals of a parallelogram, and terminated by two opposite sides, is bisected at the intersection, and itself bisects the parallelogram (Fig. 217)

(Prove $\triangle$OAP $=$ $\triangle$OCQ.)

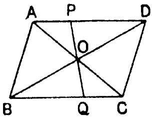


Fig. 217.

16. Prove that the bisectors of two opposite angles of a parallelogram are parallel to each other.

17. Prove that the bisectors of all the angles of a parallelogram form a rectangle (Fig. 218).

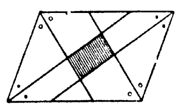


Fig. 218

18. ABCD is a parallelogram (not rectangular), and AL and CM are the perpendiculars from A and C on to the diagonal BD. Prove that ALCM is a parallelogram.

19. If ABCD is a square, and AL, BM are drawn perpendicular to any line through D, and AN is drawn perpendicular to BM or BM produced, prove that ALMN is a square.

20. ABCD is a quadrilateral having AB parallel to DC, and AD equal, but not parallel, to BC. The angle ABC is 120°. Prove that, if AE is drawn parallel to BC meeting DC in E, the triangle ADE is equilateral.

21. AEB, BFC, CGD, DHA are the sides of a parallelogram ABCD; P is a point on the diagonal AC; EPG is parallel to AD, and HPF parallel to DC; prove that the parallelograms EBFP and HPGD are equal in area (Fig. 219)

(Note that the diagonal of a $\square^{m}$ bisects it.)

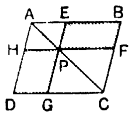


FIG. 219

22. The angle BAC of a triangle ABC is bisected and a line through C parallel to AB meets the bisecting line at D. If AB is not equal to AC, show that BD is not parallel to AC.

(Prove that CD is not equal to AB.)

23. The diagonal CA of a parallelogram ABCD is produced to E so that AE=CA. A parallelogram DAEF is constructed. Prove that BA and AF are in the same straight line.

24. ABCD is a rectangle whose diagonal AC is twice the side AB. CD is produced to P, making DP=CD. If O is the middle point of AC, prove that OP is perpendicular to AC and equal to AD.

25. An observer on the coast at O (see the rough sketch of Fig. 220) sees a vessel somewhere in the direction OA due east; 10 minutes later it is in the direction OB, and in another 10 minutes it is in the direction OC.

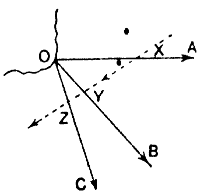


FIG. 220

If the angle AOB is 50° and the angle AOC is 80° and the speed and course of the vessel are constant, determine the true *direction* XYZ of its track by the two methods specified in (*a*) and (*b*).

(*a*) Draw an accurate figure and obtain the true direction of XYZ by arranging your ruler so that XY=YZ=1·5 inches, and then measure the angle OXY. Give briefly some justification of this experimental method.

(*b*) Justify a geometrical construction, not experimental, for finding the direction of the track XYZ, remembering that the diagonals of a parallelogram bisect one another. (*Army*.)

THEOREM 13.

The line drawn through the middle point of one side of a triangle, parallel to another side, must bisect the third side.

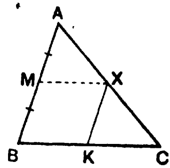


Fig. 221.

Let ABC be a triangle, M the middle point of AB, MX $\parallel$ BC.

We have to prove that X is the middle point of AC.

Construction. Draw XK $\parallel$ AB.

Proof. By definition, MXKB is a ▭^{m};

$\therefore$ XK = MB = AM. (opp. sides and hypoth.)

$\therefore$ in the two $\triangle$s AMX, XKC,

AM = XK,

$M\hat{A}X = K\hat{X}C$, (corr. angles)

$A\hat{X}M = X\hat{C}K$; (corr. angles)

$\therefore \triangle AMX \equiv \triangle XKC$; (2 angles, 1 side)

and in particular, AX = XC.

COR. 1. Because $\triangle AMX \equiv \triangle XKC$,

$\therefore$ MX = KC.

But MX also is equal to BK,

$\therefore$ BK = KC and MX = $\frac{1}{2}$BC.

COR. 2. In a trapezium ABCD if X is the middle pt. of AD, and XY $\parallel$ AB or DC, then Y must be the middle point of BC.

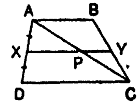


Fig. 222.

Join AC cutting XY at P.

In the $\triangle$ADC,

$\because$ AX = XD and XP $\parallel$ DC,

$\therefore$ XP = $\frac{1}{2}$DC, and P is the mid. point of AC.

Similarly in the $\triangle$ABC,

PY = $\frac{1}{2}$AB, and Y is the middle point of BC.

Also XY = XP + PY = $\frac{1}{2}$(AB + DC).

THEOREM 14.

The line joining the middle points of two sides of a triangle must be parallel to the third side, and equal to one-half of it.

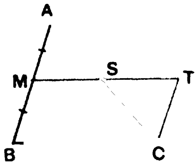


FIG. 223.

Let ABC be a triangle, M the middle point of AB, S the middle point of AC.

We have to prove that MS is || BC, and MS $= \frac{1}{2}$ BC.

Construction. Draw CT || BA, and produce MS meeting CT in T.

Proof. In the two $\triangle$s AMS, CTS,

$$A\hat{S}M = C\hat{S}T, \qquad \text{(vertically opp.)}$$

$$S\hat{A}M = S\hat{C}T, \qquad \text{(alt. angles)}$$

$$AS = CS, \qquad \text{(hypoth.)}$$

$$\therefore \triangle AMS \equiv \triangle CTS,$$

and in particular, $CT = AM = MB$.

Because CT is both equal and || BM,

$\therefore$ MTCB is a $\square^{m}$;

hence MST is || BC.

Also, $\because \triangle AMS \equiv \triangle CTS$,

$$MS = ST,$$

$$\therefore MS = \tfrac{1}{2}MT = \tfrac{1}{2}BC.$$

A **median** of a △ is a straight line drawn from any angular point to the middle point of the opposite side.

The medians of a △ are concurrent (*i.e.* pass through the same point), *and trisect one another.*

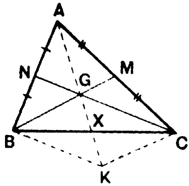


FIG. 224.

Let ABC be a triangle; BM and CN two medians meeting at G.

We have to prove that AG produced is a median.

Construction. Join AG and produce it so that GK = AG. Join CK, BK.

Proof. (i) Because AG = GK and AM = MC,

∴ MGB is ∥ CK.

Similarly, NGC is ∥ BK.

Hence BKCG is a ▱^{m}; (opp. sides ∥)

∴ BX = XC, (diags. bisect)

that is, AGX is a median;

∴ the three medians are concurrent.

(ii) Because BKCG is a ▱^{m}; ∴ GX = XK;

∴ GK = 2GX;

∴ AG = GK = 2GX,

that is, G is *a point of trisection* of AX.

Similarly, G is a point of trisection of BM and of CN.

N.B.—The point G is called the **Centroid** or sometimes the Centre of Gravity of the △.

THEOREM 15.

If three or more parallel straight lines make equal intercepts on any transversal, they make equal intercepts on any other transversal.

Let AP, BQ, CR, DS, etc., be a series of parallel straight lines cutting off from AD intercepts such that AB = BC = CD.

Let PQRS be any other transversal.

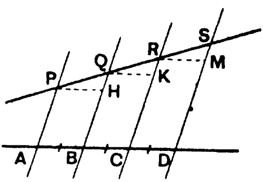


FIG. 225.

We have to prove that PQ = QR = RS.

Construction. Draw PH, QK, RM parallel to AD; they are necessarily parallel to each other.

Proof. By definition, PB, QC, RD are parallelograms;

$\therefore$ PH = AB; QK = BC; RM = CD. (opp. sides)

And thus PH = QK = RM.

In the two $\triangle$s PQH and QRK,

(i) $\hat{QPH} = \hat{RQK}$, (corr. angles)

(ii) $\hat{PQH} = \hat{QRK}$, (,,)

(iii) PH = the corresponding side QK, (proved above)

$\therefore$ $\triangle$PQH $\equiv$ $\triangle$QRK, (2 angles, 1 side)

and in particular, PQ = QR.

Similarly, by comparing the $\triangle$s QRK, RSM, we can prove

QR = RS; $\therefore$ PQ = QR = RS.

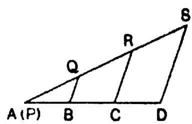


FIG. 226.

Note if A and P coincide, this theorem justifies the construction given on page 47 for the division of a line AS into an equal number of parts.

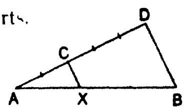


FIG. 227.

COR. To divide a line AB in a given ratio such as 2 : 3, proceed as if it is required to divide it into (2 + 3), *i.e.* 5 equal parts; then join the last point of division D to B, and through C the extremity of the second section draw a line CX $\parallel$ DB.

EXERCISE 23.

1. Draw a $\triangle$ABC, with AB=4, BC=5, CA=2·5 cms. Bisect BC at D and join AD. Measure the angles BAD, DAC; are they equal?

2. Draw a line AB 5 ins. long and divide it into 7 equal parts. Measure one of these parts.

3. Draw a quadrilateral with two sides parallel, making the lengths of these sides 3·4 and 5·8 cms. Bisect each of the other sides; measure the length of the line joining the points of bisection; and show by the drawing that it looks parallel to two sides.

4. Draw a $\triangle$ABC with BC=4·8, CA=5·2, AB=6·4 cms. Draw the medians and measure them. Verify that the centroid is the point of trisection of the medians.

5. Repeat Qn. 4, taking BC=7·8, CA=4·2, AB=5·4 cms.

6. Draw a triangle PQR, having PQ=5, QR=6, RP=7 cms. Construct a triangle ABC such that Q and R shall be the middle

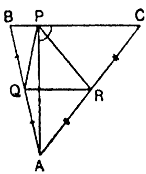


FIG. 228.

points of the sides AB and AC, and that the foot of the perpendicular from A to BC shall be P. Measure AC.

7. Prove that the lines joining the middle points of the sides of a triangle divide the triangle into four congruent triangles.

8. A quadrilateral is such that its diagonals are perpendicular. Show that the quadrilateral formed by joining the middle points of its sides is a rectangle.

9. Show that the lines which join the middle points of the opposite sides of a quadrilateral bisect each other.

(Remember that the diagonals of a ▱^{m} bisect one another.)

10. Show how to construct a triangle, if you know the three middle points of the sides.

11. ABCD is a parallelogram, H and M the middle points of AB and CD. Prove that CH and AM trisect DB.

12. Prove that any two medians (BE, AD) are together greater than the third median (CF).

(Make OG = OA.)

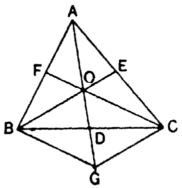


FIG 229.

13. In the figure of Qn. 12, prove that O is also the centroid of $\triangle$DEF.

14. Show how to construct a triangle ABC given the length of BC and also the lengths of the medians bisecting AB and AC.

15. AD is a given straight line which has to be trisected. Prove the following construction: Through A draw any line AK and produce it to H, making HK = AK. Join HD and produce it to Q, making QD = DH. Then QK will meet AD at a point of trisection.

16. In a right-angled $\triangle$ABC, if M is the middle point of the hypotenuse BC, prove that

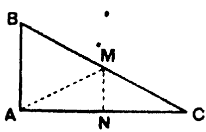


FIG. 230.

$$MA = MB = MC.$$

Conversely, if in a $\triangle$ABC, M is the middle point of BC and

$$MA = MB = MC,$$

prove that $\quad B\hat{A}C = 90°.$

(Draw MN $\parallel$ BA and prove $\triangle AMN \equiv \triangle CMN$.)

LOCI.

DEF. The path traced out by a point which moves subject to certain conditions is called the **Locus** of the point.

Ex. 1. If a point moves so that it is always 1 inch above a given line AB, its locus will be PQ, a parallel straight line 1 inch away.

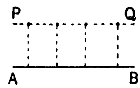


FIG. 231.

This line PQ may be obtained by drawing a number of lines 1 inch long, each perp. to AB.

Ex. 2. If a point moves with the condition that it must always be at a constant distance 2 cms from a fixed point, the locus is a circle of 2 cms. radius with the fixed point as centre.

Ex. 3. If a dog is attached to a chain, at the other end of which is a ring sliding on a fixed horizontal pole, the

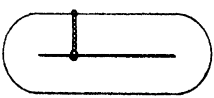


FIG. 232

dog's locus is anywhere between two parallel lines and two semi-circles, as in Fig. 232.

The shape and position of the locus can often be determined by geometrical reasoning; but in order to familiarize the student with the idea of a locus, we shall begin with examples which are mainly to be worked out by plotting a number of points satisfying the given conditions and joining these points by a straight line or curve. In most cases the locus will be a straight line or part of a circle, but the student must expect to find other curves as well.

EXERCISE 24.

1. Plot a number of points 3·5 cms. from a given line XY, and thus determine the locus of a point P which moves so as always to be 3·5 cms. from XY.

2. Draw any two parallel lines AB and CD, then draw any number of transversals such as PQ; mark the middle point O of each line PQ, and determine the locus of O (Fig. 233)

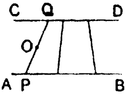


Fig. 233

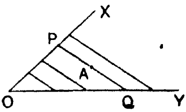


Fig. 234

3. Draw two lines OX, OY so that $X\hat{O}Y = 40°$. Draw a number of parallel lines such as PQ, so that $O\hat{P}Q = 110°$, in each case mark A the middle point of PQ, and thus find the locus of A (Fig. 234).

4. Assuming that if a circle touches a straight line AB, then OC, the radius to the point of contact, is at right angles to AB, suppose a wheel rolls along a horizontal road AB, what is the locus of the centre of the hub? (Fig 235).

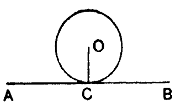


Fig. 235.

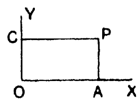


Fig. 236.

5. Let $X\hat{O}Y = 90°$. Draw a number of rectangles like APCO with OA=2OC.
Construct the locus of P (Fig. 236).

6. Repeat Qn. 5 making OA=3OC

7. With Fig. 236, let P move so that

PA+PC=8 cms., i.e. OC+OA=8 cms.

Construct various positions of P so that

OA=0, 1, 2 ... 8 cms respectively, and draw the locus of P.

8. With Fig 236, let P move so that

PC − PA=5 cms, i.e. OA − OC=5 cms.

Plot the locus of P.

9. Draw a number of isosceles triangles ABC on the same base AB; plot the locus of C.

10. Draw a $\triangle ABC$ with $AB=3$ cms., $AC=4{\cdot}2$ cms., $BC=5{\cdot}3$ cms. Draw a number of lines, such as AX, and mark P the middle point of AX.

What is the locus of P ?

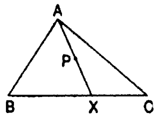


Fig. 237.

11. Draw a number of isosceles $\triangle$s ABC on the same base AB; in each case, produce AC to P so that $AC=CP$. What is the locus of P ?

12. Plot a number of points 2 inches from O ; what is the locus of such points ?

13. What is the locus of the extremity of the hand of a clock ?

14. P moves round a circle with centre O and radius 5 cms. ; PQ is drawn vertically and is always 3 cms. long. Construct the locus of Q (Fig. 238).

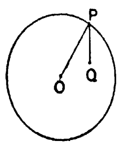


Fig. 238.

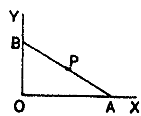


Fig. 239.

15. A rod AB, 6 cms. long, slides between two rods OX, OY, which are at right angles, and P is the middle point of AB. Plot the locus of P (Fig. 239).

16. In a piece of machinery, P moves round a circle, centre O, A moves along a line ABOC and AP is fixed in length.

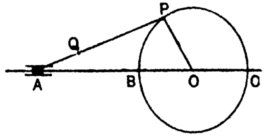


Fig. 240.

Take $OP=4$ cms., $PA=10$ cms., and construct the locus of Q, a point on AP such that $AQ=2$ cms.

(Let P move through a semi-circle from B to C.)

17. A rod AB of length 6 cms. slides between two rods OX, OY, which are at right angles. If P is a point on AB so that AP=2 cms., plot the locus of P.

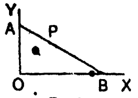


FIG. 241.

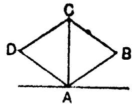


FIG. 242.

18. A rhombus ABCD formed of 4 jointed rods, each of length 5 cms., is placed so that AC is always vertical. As AC alters in length from 0 to 10 cms., draw the locus of B (Fig. 242).

19. Draw a diameter AOB of a circle of 4·3 cms. radius. Draw a number of semi-chords, such as PQ, each perpendicular to AB. If M is the middle point PQ, plot the locus of M.

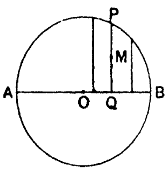


FIG. 243.

20. Two points A and B are 6 cms. apart, and a point P moves so that PA + PB = 10 cms. Draw a number of positions of P by taking

PA = 2, 3, 4 ... 8 cms.,

and PB = 8, 7, 6 ... 2 cms.,

and thus find the locus of P.

21. A point P moves so that the product of its distances from two lines OX, OY, which are at right angles, is always 10. Plot the locus of P.

In order to prove that a certain path is the locus of a point moving under certain conditions, it must be shown

(i) that any point which satisfies the conditions must lie on the path,

(ii) that any point on the path satisfies the conditions;

and consequently a formal proof necessitates the proof of a theorem and its converse.

It may be noticed further, that any point not on the path does not satisfy the conditions.

THEOREM 16.

The locus of points equidistant from two fixed points is the perpendicular bisector of their join (*i.e. the line joining them*).

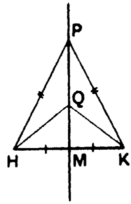


FIG. 244.

Let H, K be the two fixed points.

(i) If P is a point on the locus, so that PH = PK, we have to prove that P lies on the perpendicular bisector of HK.

Construction. Bisect HK at M and join PM.

Proof. In the △s PHM, PKM,

PH = PK,

HM = MK, (constr.)

PM is common ;

∴ △PHM ≡ △PKM, (3 sides)

and in particular, $P\hat{M}H = P\hat{M}K = 90°$;

∴ P lies on the line bisecting HK at right angles.

(ii) Again, if Q is any point on the perpendicular bisector of HK, we have to prove that QH = QK.

Proof. In the △s QHM, QKM,

HM = MK, (HK bisected)

$Q\hat{M}H = Q\hat{M}K$, (each 90°)

QM is common ; (2 sides, incl. angle)

∴ △QHM ≡ △QKM,

and in particular QH = QK.

Thus PM (produced both ways) is the required locus.

This result may be obtained by the Principle of Symmetry. Since PM is an axis of symmetry for the two points H and K, it follows that any point P on this axis is equidistant from H and K, and any point off the axis is nearer either to H or K. (See page 62).

THEOREM 17.

The locus of points equidistant from two given straight lines is the pair of straight lines which bisect the angles between the two given lines.

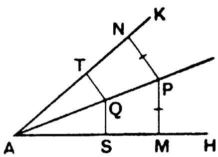


FIG. 245

Let AH, AK be the two given straight lines.

(i) If P is a point on the locus, so that perp. PM = perp. PN, we have to prove that

$$\hat{PAN} = \hat{PAM}.$$

Construction. Join AP.

Proof. In the right-angled $\triangle$s PAN, PAM,

PN = PM,

hypotenuse AP is common ;

$\therefore$ $\triangle$PAN $\equiv$ $\triangle$PAM,

and in particular, $\hat{PAN} = \hat{PAM}$,

i.e. P lies on the bisector of the angle HAK.

(ii) Again, if Q is any point on the bisector of the angle HAK, we have to prove that

perp. QS = perp. QT.

Proof. In the right-angled $\triangle$s QAT, QAS,

$$\hat{QAT} = \hat{QAS},$$

hypotenuse AQ is common ;

$\therefore$ $\triangle$QAT $\equiv$ $\triangle$QAS,

and in particular, QT = QS.

Thus PA (produced both ways) is part of the required locus.

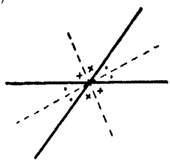


FIG. 246.

Similarly it may be shown that the bisector of the supplementary angle between the two lines is also part of the locus.

By the Principle of Symmetry, AH and AK are symmetrically situated with respect to the axis AP, or PA produced, and therefore any points such as T, S are equidistant from the axis AP.

Similarly, the bisector of the obtuse angle is an axis of symmetry.

Intersection of Loci.

If a point moves so that it satisfies more than one condition, then by noting its locus if the first condition only is fulfilled, and also its locus if the second condition only is fulfilled, the intersection of these loci determines the actual position or positions of the point.

Ex. *Given a* △ABC, *find the position of a point which is equidistant from* AB *and* AC *and a given distance above* BC.

(i) If the point is to be equidistant from AB and AC, it must be on AX, the bisector of the angle BAC.

(ii) Draw a line YZ at the given distance from BC; then the point must also be on YZ.

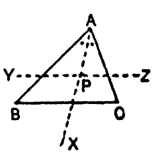


FIG. 247.

∴ the point required must be P, which is the point of intersection of AX and YZ.

EXERCISE 25.

1. PQ is any line of indefinite length, A and B fixed points not in PQ. Find a point in PQ equidistant from A and B.
(Hint: Construct the locus of points equidistant from A and B.)

2. ABC is a triangle inside a circle. Show how to find a point on the circle equidistant from B and C.

3. ABC is a triangle inside a circle. Show how to find a point on the circle equidistant from the lines AB and AC.

4. Draw an angle of 40°. Draw the locus of all points between the arms 1 cm. from one of the arms. Also draw the locus of all points between the arms 2 cms from the other arm. Hence find a point 1 cm. from one arm and 2 cms. from the other
Measure the distance of this point from the angular point.

5. Draw an angle of 50°. Find a point within the arms of the angle 3 cms. from each arm. Measure the distance of this point from the angular point.

6. Draw a triangle $a=7$, $b=6$, $c=5$; and find a point P in AC equidistant from B and C. Measure PC

7. Draw a triangle $a=5$, $b=10$, $c=7$. Find a point K in BC, equidistant from AB and AC. Measure these equal distances, and also KB and KC.

8. Draw a triangle, sides 5, 6, 7, and find a point equidistant from all three sides. Measure these equal distances.

9. AOB is a diameter of a circle, centre O and radius 3 cms. Find a point on the circumference equidistant from O and B and measure its distance from A

10. AB and CD are two parallel lines 3 inches apart; a point P is 2 inches from AB and 1 inch from CD; draw its locus. If the point P is also 3 inches from a given point X on AB, find its position.

11. Show how to find two points, each equidistant from two given points, and each 1 cm. from a given line

12. Show how to find a point equidistant from two points and at a given distance from a third point.
This problem is not always possible, state in words when you think it is impossible.

13. Draw a $\triangle$ABC with AB=2·5 ins., BC=2 ins, AC=1·4 ins. Show how to find all the points that are 0·6 in. from BC (or BC produced), and equidistant from AB and AC (or AB produced and AC).
(Make BC more or less parallel to the long edges of your paper.)

14. Find the points in the line AB which are equidistant from the two intersecting lines X and Y (Fig. 248).

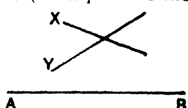


FIG. 248.

15. Draw a $\triangle$ABC on a base BC of length 5 cms., so that its height is 4 cms., and the side BA=4·5 cms. Measure the two values of AC.

16. Draw any triangle ABC. Construct the locus of points equidistant from AB and AC, also the locus of points equidistant from B and C. Hence find the point which is equidistant from AB and AC, and also equidistant from B and C

17. In a $\triangle$ABC, BC=5·4 cms, BA=4·3 cms, CA=3·7 cms., what is the locus of points which are equidistant from B and C, and less than 1 inch from A.

18. Essential parts of a mechanism for fastening a gate are shown in Figs. 249 and 250, which are not drawn to scale. It consists of 3 rods, AB, BC, CD, free to move about pivots at B and C The rods are in the same vertical plane. AB is vertical when the gate is closed; BC turns on a fixed pivot at F, the free end of CD shoots into a slot of the gate-post and so fastens the gate Certain parts of the gate serve as guides for the rods, but these you need not consider

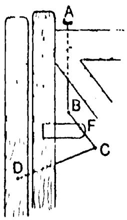


FIG. 249

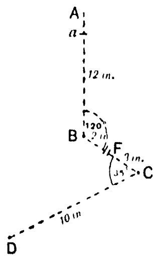


FIG 250.

To open the gate, A is pushed down to *a*, and the downward motion thus given to B causes BC to rotate about F, and thus the end D of the rod CD is withdrawn from its slot in the gate post.

Draw the mechanism to a scale of 1 cm to 1 inch, placing AB in the middle of the page, showing first the position of the rods for the closed gate with *dotted* lines, and then, on the same diagram, their position for the opened gate with *continuous* lines, given that the rod AB is 12 ins long, BC 5 ins. (BF being 2 ins and FC 3 ins), CD 10 ins, and note that *in the first position* $\angle$ABC is 120° and $\angle$BCD 35°.

For the second position suppose A to have been pushed down through a vertical distance of 1½ ins. to *a*, and the rod CD to be so supported that D's final position is 1½ ins. further to the right than before, but not on the same level, and remember that all parts of the mechanism are movable except the pivot F, which is fixed throughout. Denote the new positions of the rods by *ab*, *bc*, *cd*, indicate the paths of B and C, and state clearly how you determine the final positions of B and D. Show all lines of construction, and mark the various dimensions on your figure. (*Army*.)

AREAS.

It is not always easy or possible to tell by inspection which of two areas is the larger, nor is it possible to say how many times a smaller area is contained in a larger, nor even if two areas are equal.

We could not tell by inspection that the two figures, 251 and 252, are equal in Area, but Geometry decides the fact that they are equal.

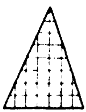


Fig. 251.

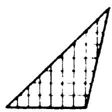


Fig. 252.

An **approximate practical** method of estimating **Area** is to draw the figures on **Squared Paper** and to count up the small squares, neglecting any portion of a small square which looks less than half and counting as a whole any portion which looks more than a half.

Geometry uses the idea of this method; it first selects a square to be taken as the **Standard** or **Unit Square**, and then proceeds to find methods by which the actual number of squares contained in a figure may be calculated.

In actual practice in surveying, the *chain* (= 100 *links*) is frequently used as the unit of length, the chain containing 22 yds.

Thus, 1 sq. chain $= 22^2$ sq. yds. $= 484$ sq. yds.

$= \frac{1}{10}$ acre.

Figures which have the same area are often called **Equivalent Figures.**

THEOREM 18.

The area of a rectangle is measured by the product of the measures of its sides.

Take a rectangle, sides 5 and 3 cms., Fig. 253.

Draw parallels as in the figure.

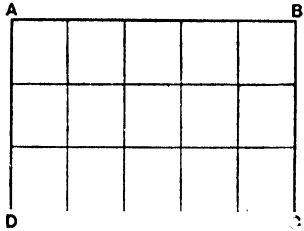


FIG. 253.

The student will have no difficulty in seeing that the small quadrilaterals are squares with each side 1 cm.

In each row there are 5 small squares and there are 3 such rows,

$\therefore$ the area of the rectangle is 3×5 or 15 sq. cms.

A similar result holds for any numbers which may be chosen.

Thus, **area of a rectangle = length × breadth.**

This may be put algebraically :

the area of a rectangle of sides a and b cms.

$= ab$ sq. cms.

In the above figure the rectangle is said to be ***contained*** by any two adjacent sides such as AB and AD, and the area may be written AB . AD.

THEOREM 19.

The area of a parallelogram is equal to the area of a rectangle on the same base and between the same parallels.

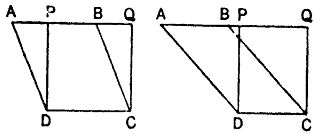


FIG. 254.

Let ABCD and PQCD be a parallelogram and rectangle on the same base DC and between the same parallels DC and AQ, so that they have the same altitude or height.

We have to prove that

area of ▱^{m} ABCD = area of rect. PQCD.

Proof. In the two right-angled △s DAP, CBQ,

AD = BC, (opp sides of ▱^{m})

DP = CQ, („ „ rect.)

$\hat{APD} = \hat{BQC}$, (rt. angles)

∴ △DAP ≡ △CBQ,

and in particular, are equal in area.

∴ fig. ADCQ − △CBQ = fig. ADCQ − △DAP,

i.e. area of ▱^{m} ABCD = area of rect. PQCD.

COR. 1. *The area of a ▱^{m} is measured by the product of the measures of its base and altitude.*

Area of ▱^{m} ABCD = area of rect. PQCD

= DC . DP

= base of ▱^{m} × alt. of ▱^{m}.

COR. 2. *Parallelograms on the same or equal bases and of the same altitude are equal in area* (Fig. 255).

COR. 3. *If two parallelograms of the same area have the same or equal bases, their altitudes are equal.*

(See Cor. 3, Theorem 20.)

FIG. 255.

THEOREM 20.

The area of a triangle is one-half of the area of a rectangle on the same base and between the same parallels.

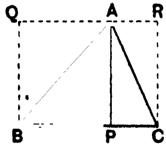


FIG. 256.

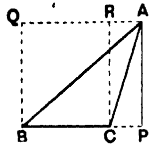


FIG. 257.

Let ABC be any triangle.

Construction. Draw AP an altitude perpendicular to BC. Draw QAR parallel to BC and BQ, CR $\perp^{\text{ar}}$ to BC.

Then the $\triangle$ABC and rect. QBCR are on the same base BC and between the same parallels BC and QR.

We have to prove that

$$\text{area of } \triangle ABC = \tfrac{1}{2} \text{ area of rect. } QBCR.$$

Proof. The diagonal of a parallelogram divides the parallelogram into two congruent triangles,

$$\therefore \quad \triangle ABP = \tfrac{1}{2} \text{ rect. } QBPA,$$

$$\text{and} \quad \triangle APC = \tfrac{1}{2} \text{ rect. } APCR,$$

$\therefore$ by addition in Fig. 256 and subtraction in Fig. 257,

$$\text{area of } \triangle ABC = \tfrac{1}{2} \text{ area of rect. } QBCR.$$

COR. 1. *The area of a $\triangle$ is measured by one-half the product of the measures of its base and altitude.*

Area of $\triangle ABC = \tfrac{1}{2}$ area of rect. QBCR

$= \tfrac{1}{2}BC \,.\, BQ = \tfrac{1}{2}BC \,.\, PA$

$= \tfrac{1}{2}$ base $\times$ height.

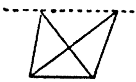


FIG. 258.

COR. 2. *Triangles on the same or equal bases and of the same altitude are equal in area* (Fig. 258).

COR. 3. *If two triangles of the same area have the same or equal bases their altitudes are equal* (Fig. 259).

In particular, if two triangles of the same area stand on the same side of the same base, or of equal bases in the same straight line, they are between the same parallels.

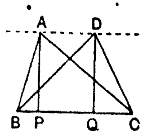


FIG. 259.

(i) Area of $\triangle ABC$ = area of DBC,

$\therefore$ BC . AP = BC . DQ,

$\therefore$ AP = DQ.

(ii) Since AP and DQ are both perp to BC,

$\therefore$ AP $\parallel$ DQ ;

but AP = DQ,

$\therefore$ AD $\parallel$ BC, since AD and BC join the extremities of equal and parallel lines.

If any number of triangles of equal area stand on the same side of a common base, it follows that the *locus of the vertices* is a straight line parallel to the base

THEOREM 21.

If a parallelogram and a triangle stand on the same base, and are between the same parallels, the area of the parallelogram is double that of the triangle.

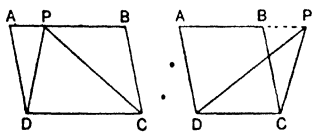


FIG. 260.

Let ABCD and DPC be a parallelogram and triangle on the same base DC and between the same parallels DC and AB or AP.

We have to prove that

area of $▱^m$ ABCD = area of $\triangle$ DPC.

Proof. Area of $▱^m$ ABCD = DC × altitude,

Area of $\triangle$DPC = $\frac{1}{2}$DC × altitude,

and since the altitudes are the same,

$\therefore$ area of $▱^m$ ABCD = 2 × area of $\triangle$DPC.

THE AREA OF A TRAPEZIUM.

DEF. *A* **trapezium** *is a four-sided figure with two of its sides parallel.*

Let ABCD be a trapezium with AB and DC parallel. Join BD.

Draw DM and BN perp. to BA produced and DC respectively; since DM and BN are perp. distances between the same two parallels,

$\therefore$ DM = BN = h (suppose).

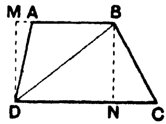


FIG. 261.

Area of trapezium $= \triangle ABD + \triangle BDC$

$= \frac{1}{2}AB \,.\, h + \frac{1}{2}DC \,.\, h$

$= \frac{1}{2}(AB + DC)h$;

i.e. area of trapezium = *the product of half the sum of the parallel sides and the perp. distance between them.*

NUMERICAL EXAMPLES ON AREA.

The area of a figure such as Fig. 262 is found by dividing it into rectangles.

$A = 3 \times 5 = 15$; $C = 6 \times 3 = 18$;

$B = 5 \times 2 = 10$; $D = 4 \times 10 = 40$;

$E = 2 \times 3 = 6.$

$\therefore$ area $= 15 + 10 + 18 + 40 + 6$

$= 89$ units of area.

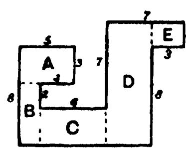


FIG. 262.

The area of a figure such as Fig. 263 is found by dividing it into triangles.

Suppose AC = 6; AD = 10; AE = 8; the lengths of the altitudes are marked in the figure.

$\triangle ABC = \frac{1}{2} \,.\, 3 \,.\, AC = 9,$

$\triangle ACD = \frac{1}{2} \,.\, 5 \,.\, AD = 25,$

$\triangle ADE = \frac{1}{2} \,.\, 6 \,.\, AE = 24,$

$\triangle AEF = \frac{1}{2} \,.\, 4 \,.\, AE = 16$;

$\therefore$ area $= 9 + 25 + 24 + 16$

$= 74$ units of area.

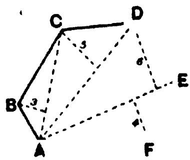


FIG. 263.

FIELD BOOK.

It is often convenient to measure an area by dividing it up into right-angled triangles and trapeziums from records given in a table such as the following:

Yards.

	to D	
	400	
	312	60 to E
to C 100	250	
	105	90 to F
to B 70	60	
	from A	

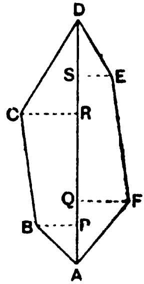


FIG. 264

It will be seen from the figure that

$$AD = 400,$$
$$AS = 312, \quad SE = 60,$$
$$CR = 100, \quad AR = 250,$$
$$AQ = 105, \quad QF = 90,$$
$$BP = 70, \quad AP = 60;$$

Now area of

$\triangle ABP = \frac{1}{2}AP \times BP \quad = \frac{1}{2} \times 60 \times 70 = 2100$ sq. yds.

trap^m $BPRC = \frac{1}{2}PR(BP + CR) = \frac{1}{2} \times 190 \times 170 = 16150$ sq. yds.

$\triangle CRD = \frac{1}{2}CR \times RD \quad = \frac{1}{2} \times 100 \times 150 = 7500$ sq. yds.

$\triangle DSE = \frac{1}{2}DS \times SE \quad = \frac{1}{2} \times 88 \times 60 = 2640$ sq. yds.

trap^m $ESQF = \frac{1}{2}QS(SE + QF) = \frac{1}{2} \times 207 \times 150 = 15525$ sq. yds.

$\triangle AFQ = \frac{1}{2}AQ \times FQ \quad = \frac{1}{2} \times 105 \times 90 = 4725$ sq. yds.

$\therefore$ area of field ABCDEF = 48640 sq. yds.

CONSTRUCTION 6.

To construct a triangle equal in area to a quadrilateral.

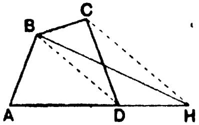


FIG. 265

Let ABCD be a quadrilateral. It is required to construct a triangle of the same area.

Construction Join BD and draw CH ∥ to BD meeting AD produced at H. Join BH.

Proof. The $\triangle$s BDC and BDH are on the same base BD and between the same ∥ BD and CH; they are therefore equal in area.

Hence $\triangle ABD + \triangle BCD = \triangle ABD + \triangle BDH$;

$\therefore$ area of ABCD $= \triangle ABH$.

To construct a polygon equal in area to a given polygon, but having one side fewer.

Let PQRSTMH be any polygon

Construction. Join any angular point M to the next but one S.

Draw TA ∥ SM meeting HM produced at A.

Proof. Exactly as in the last construction,

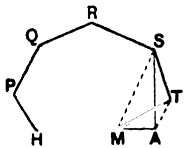


FIG. 266.

$\triangle SMA = \triangle SMT$;

$\therefore$ area of PQRSMH + $\triangle SMA$ = PQRSMH + $\triangle SMT$;

$\therefore$ area of PQRSAH = area of PQRSTMH;

thus we have constructed a 6-sided polygon equal to one with 7 sides.

By reducing the new polygon to one of 5 sides, and so on, we can ultimately obtain a triangle having the same area as the polygon.

EXERCISE 26.

1. Draw a triangle ABC where AB = 6 cms., CA = 6·2 cms. and BC = 6·7 cms.

Draw perps. AD, BE, CF to the opposite sides.

Calculate the area of $\triangle$ ABC by finding the values of

$$\tfrac{1}{2}AB \times CF, \quad \tfrac{1}{2}BC \times AD, \quad \tfrac{1}{2}CA \times BE$$

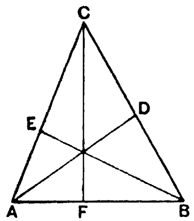


Fig. 267

2. Construct a triangle ABC on squared paper, B = 56°, $a = 7·2$, $c = 5·5$. Find the area by counting the small squares. Measure the altitude from A, and hence find the area by a different method.

3. Construct a triangle ABC having an area of 8 sq. cms., the side BC 4 cms. long, and the side AB 4·5 cms. long. Measure the length of AC in the two cases.

4. ABC is a triangle, in which AB = 2·4 inches, BC = 2·2 inches, and the perp. from C to AB is 1·5 inches. Construct the triangle, measure AC in both cases and calculate the area.

5. Construct a ▭^{m} on squared paper, sides 4 and 6 and one angle 54°. Find the area by counting small squares. Measure the height of the ▭^{m} from a side 6 and again find area.

6. Draw a ▭^{m} with sides 4·3 and 3·4 cms. and contained angle 52°. Find its area.

7. Draw a ▭^{m} with two sides 3·2 and 2·7 inches and a diagonal 3·8 inches. Find its area.

8. Draw a ▭^{m} with sides 5·6 and 4·7 cms. and contained angle 44°. Then draw a rhombus equal in area and with all its sides 5·6 cms., and measure its diagonals.

9. Draw a rhombus with sides 5·4 cms. and diagonal 3·9 cms. Find its area.

10. Draw a ▭^{m} with diagonals 3·8 and 5·8 cms. and the angle between them 54°. Find its area.

11. Draw an isosceles $\triangle$ on a base of 4·2 cms. and vertical angle 40°. On the same base draw another triangle of equal area and with one of its base angles 50°, and measure its sides.

12. One set of parallel lines are 4·5 cms. apart and another pair 3·3 cms. apart; if they cut at 48°, find the area of the parallelogram so formed.

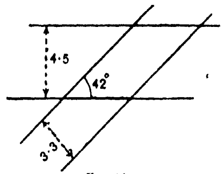


Fig. 268.

13. Draw a ▱^{m} of area 9·4 sq. cms. on a base of 4·7 cms. with one of its diagonals 3·2 cms. Measure the other diagonal.

(Calculate the height of the ▱^{m}.)

14. Draw a ▱^{m} of area 35 sq. cms., and sides 6 cms. and 7 cms. respectively. Measure the diagonals.

15. Draw a triangle on a base of 4·7 cms. and area 20·21 sq. cms., with one of the base angles 52°. Measure the lengths of the other two sides of the triangle.

16. Draw a trapezium ABCD in which AB = 3 cms., AD = 5·2 cms., DC = 7 cms and $\hat{ADC}$ = 60°.

Find the area.

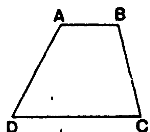


Fig. 269.

17. Calculate the area of the symmetrical T-shaped figure with the given dimensions in cms. (Fig. 270).

18. Calculate the area of the L-shaped figure with the given dimensions in cms. (Fig. 271).

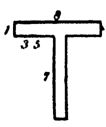


Fig. 270.

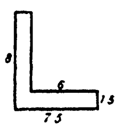


Fig. 271.

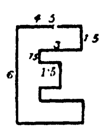


Fig. 272.

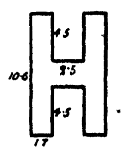


Fig. 273.

19. Calculate the area of the symmetrical E-shaped figure with the given dimensions in cms. (Fig. 272).

20. Calculate the area of the H-shaped figure with the given dimensions in cms. (Fig. 273).

21. Calculate the area of Fig. 274, the dimensions being given in cms.

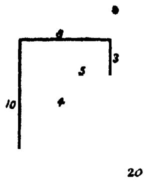


FIG 274

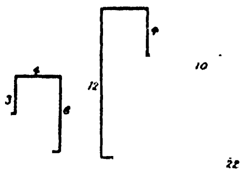


FIG. 275

22. Calculate the area of Fig. 275 in sq. cms.

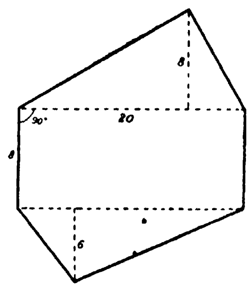


FIG. 276.

23. Calculate the area of Fig. 276 in sq. inches.

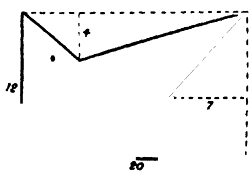


FIG. 277.

24. Calculate the area of Fig. 277 in sq. inches.

25. Calculate the areas of fields taken from

	Yards	
	60	
14	48	
	32	10
8	15	
	10	16

	Metres.	
	50	
18	42	
6	30	
	20	20

	Links.	
	420	
	340	160
120	260	
	120	100

26. Construct a parallelogram having its diagonals 10 cms. and 9 cms., and containing an area of 40 sq. cms. Measure its sides.

(Calculate the perp. AB and thus first construct a quarter of the ▱m, Fig. 278).

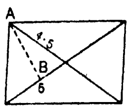


Fig. 278

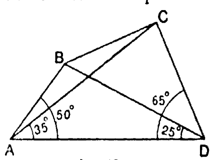


Fig. 279

27. ABCD (Fig. 279) is a rough sketch of a level field on a scale of 1 in. to 50 yards. AD = 3·7 inches, and the angles marked are given. Draw the figure and calculate the area of the field in sq. yards. (*Army.*)

28. Construct a quadrilateral ABCD having AB = 7 cms., AD = 10 cms., ∠ABC = 120°, ∠BAD = 80°, ∠ADC = 70°.

Find the area of the quadrilateral.

29. Construct a quadrilateral ABCD from the following data: AB = 3 ins., BC = 2 ins., AC = 3·6 ins., ∠BAD = 104°, BD = 4 ins.

Determine the area of this quadrilateral correct to the nearest tenth of a square inch.

(First construct △ABC.)

30. Construct a triangle having sides 6·4, 8·6, 11·2 cms. long. On the longest side construct a parallelogram equal in area to the triangle and having one of its angles equal to 70°.

Measure the shorter sides of the parallelogram.

31. Construct a quadrilateral ABCD from the following data: AB is parallel to DC, AB = 10·8 cms., CD = 5·6 cms., AD = 6·3 cms., and ∠B = 60°. Find its area.

(Draw CE parallel to DA meeting AB at E. Construct △CEB.)

32. Construct a quadrilateral ABCD having AB = 3·9 ins., BC = 3 ins., CD = 2·7 ins., $\hat{ABC}$ = 60°, $\hat{BCD}$ = 97°.

Construct a rectangle of equal area on AB as one side, and measure its height.

(Reduce quad. to a △ with AB for one side, then convert △ into a rectangle.)

33. Draw a pentagon ABCDE with AB=4, BC=3·2, CD=4·3, DE=3·4, EA=2·7 cms., $\hat{BAE}$=110°, $\hat{ABC}$=115°. Reduce it to a triangle and find its area.

34. Construct a convex quadrilateral LMXY, having LM=10 cms., MX=7·3 cms., XY=6·5 cms., LY=5·4 cms., and the angle MLY=55°. Construct a triangle of equal area, and find its area in square centimetres.

35. Construct a rectangle, with one side 2·8 cms., equal in area to a rectangle of sides 4·3 and 3·4 cms. Measure the other side (Use the method of Ex. 22, 21, Fig. 280.)

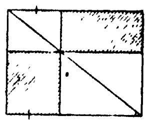


Fig. 280.

36. Construct a parallelogram, with one side 4·2 cms. and one angle 65°, equal in area to another parallelogram with sides 3·4 cms. and 3·9 cms., and contained angle 65°. Measure the other side.

37. Draw a quadrilateral ABCD with AB=3·2 cms., BC=4·9 cms., DA=4·2 cms., $\hat{A}$=115°, $\hat{B}$=100°. Reduce it to an equivalent triangle, then to a rectangle, and finally to another rectangle with one side 5·4 cms. Measure an adjacent side of the rectangle.

38. Draw a pentagon ABCDE with AB=3·2, AE=4·2, EB=5·4, EC=4·3, BC=2·3, ED=3·7, CD=2·8 cms. Reduce it to an equivalent triangle and thence into a parallelogram with one angle 72°.

39. Draw a △ABC with a=5·4, b=3·7, c=4·2 cms. Draw an equivalent △DEC, with one side DC=4·8 cms. Measure DE, EC. (Note that the triangles have a common angle at C, Fig. 281.)

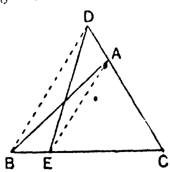


Fig. 281.

40. Draw a △ABC with a=6·7, b=4·3, c=6·4 cms. Draw an equivalent △DBC with one side DC=5·7 cms. Measure the two values of BD (Fig. 282.)
(Note that the triangles have a common side BC.)

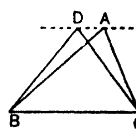


Fig. 282

41. Draw a △ABC with a=6·4, b=4·7, c=5·2 cms., and then construct an equivalent right-angled △ with one side 4 cms. Measure its hypotenuse.
(Convert the △ABC into an equivalent △BCD with CD=4 cms., and then draw right-angled △CDE equivalent to △CDB.)

42. Draw a $\triangle ABC$ with $a=5\cdot4$, $b=4\cdot7$, $c=5\cdot1$ cms. On the same base BC construct an equivalent $\triangle BCD$ with $CD=5\cdot2$ cms. Then on CD as base construct another equivalent isosceles $\triangle CDE$. Measure the sides of the isosceles $\triangle CDE$.

(We have thus drawn, on a given base, an isosceles triangle equivalent to a given triangle.)

43. Draw an equilateral $\triangle ABC$ with sides 6 cms.; construct an equivalent $\triangle ACD$ with $AD=7\cdot5$ cms. On AD as base construct a $\triangle ADE$ equal in area to $\triangle ACD$ and with $A\hat{D}E=55°$. Measure AE

(We have thus drawn a triangle, with one side and one adjacent angle of given magnitudes, equivalent to a given equilateral triangle.)

44. Draw two lines OX, OY inclined at an angle of 66°; along OX measure $OA=1\cdot5$ inches, $OB=2\cdot9$ inches; then on AB construct a triangle ABC of area $1\cdot26$ sq. inches and having its vertex C equidistant from OX and OY Measure the sides of the $\triangle ABC$.

45. ABC is a triangle, M is the middle point of AB and N of AC; prove that the triangles BMC and BNC are equal in area.

46. D, E, F are the middle points of the sides BC, CA, AB of a triangle, and G is the centroid (*i.e.* the intersection of the medians AD, BE, CF); prove that any such triangle as BGD is one-sixth of $\triangle ABC$.

47. ABCD and ABEF are two parallelograms on the same base and between the same parallels; P and Q are the mid-points of AD and BE respectively. Prove that the triangles PCD, QEF are equal in area.

48. ABCD is a quadrilateral in which AB is parallel to DC. The diagonals AC, BD intersect at O. Prove that the triangles AOD, BOC are equal in area.

49. P is any point within a parallelogram ABCD; prove that the sum of the areas of the triangles PAB, PCD is equal to the sum of the areas of the triangles PAQ, PBC (Fig. 283).

(Draw XPY parallel to AB)

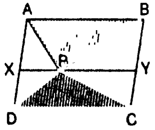


FIG. 283.

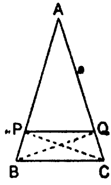


FIG. 284.

50. If ABC is an isosceles triangle and BQ, CP are drawn perp. to AC, AB, prove that PQ is parallel to BC (Fig. 284).

51. If ABC is any $\triangle$ and X, Y are the middle points of AB, AC, prove the area of $\triangle AXY=\frac{1}{4}\triangle ABC$.

52. If P, Q, R, S are the middle points of the sides of a quadrilateral ABCD, prove that the area of the $\square^m$ PQRS $= \frac{1}{2}$ area of ABCD.

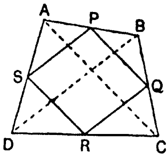


FIG. 285.

53. ABCD is a $\square^m$ and P the middle point of CD. If AP and BC, when produced, meet at Q, prove that area of $\triangle$BPQ $= \frac{1}{2}$ area of $\square^m$ ABCD (Fig. 286).

(Note that $\triangle$CPQ = $\triangle$DPA.)

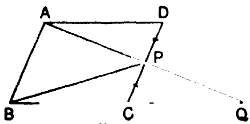


FIG. 286.

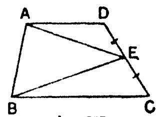


FIG. 287.

54. ABCD is a quadrilateral, in which the sides AD, BC are parallel and E is the middle point of CD. Prove that the area of the triangle AEB is half the area of the quadrilateral ABCD (Fig. 287).

55. Prove that the area of any rhombus is one-half that of a rectangle whose sides are equal in length to the diagonals of the rhombus.

56. ABC is a triangle, X a point in AB such that AX $>$ XB, and D the middle point of AC. Join XD and draw BY $\parallel$ XD, meeting AC in Y. Join XY. Prove that XY bisects the area of the $\triangle$ABC.

(Join BD and note $\triangle$XBD is equivalent to $\triangle$XDY, Fig. 288.)

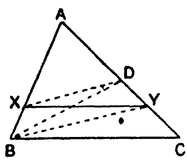


FIG. 288.

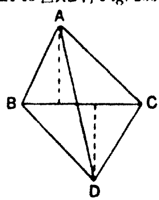


FIG. 289.

57. Two triangles ABC, DBC, equal in area, are on the same base BC and on opposite sides of it. Prove that AD is bisected by BC.

(Draw perps. from A and D to BC, Fig. 289.)

58. ABCD is a quadrilateral, K is a point in AD produced and CK is parallel to the diagonal BD; prove that the area of $\triangle$ABK is equal to the area of the quadrilateral.

59. ABCD is a trapezium in which AB is parallel to DC. DF drawn parallel to CB meets AB, produced if necessary, in F. E is the middle point of AF. Prove that the area of the △EBC is one-half that of the trapezium.

60. ABCD is a square; P is any point in AD; CP and BA are produced to meet at Q, and DQ is drawn. Prove that the triangles QPD, APB are equal in area.

61. ABCD is a ▱^{m} and through A a straight line AQP is drawn, cutting BC in Q and DC produced in P. Prove that the △s BQP and DQC are equal in area.

62. Bisect a quadrilateral by a line drawn through one of the angular points.
(Reduce the quad. to a triangle.)

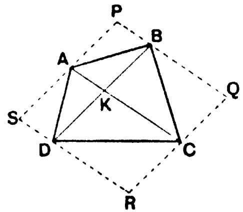


FIG. 290

63. Prove that the area of a quadrilateral is equal to one-half a parallelogram of sides equal and parallel to the diagonals of the quadrilateral (Fig. 290).

THEOREM OF PYTHAGORAS.

(i) If a right-angled △ABC is constructed with AC = 3 cms. AB = 4 cms. and $C\hat{A}B = 90^\circ$, then, on measurement, it will be found that BC = 5 cms.

Now $\quad 5^2 = 25$ and $3^2 + 4^2 = 25$;

$$i.e.\ BC^2 = AB^2 + AC^2.$$

(ii) If any right-angled triangle is constructed on squared paper and the squares on AB, BC, CA drawn, then, on counting the small squares, we again find that

$$BC^2 = AB^2 + AC^2.$$

(iii) If any right-angled triangle and the corresponding squares are drawn on card-board, and the three squares cut out and weighed, we find that the square on BC weighs the same amount as the sum of the squares on AB and AC.

THEOREM 22.

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides. (*Pythagoras' Theorem.*)

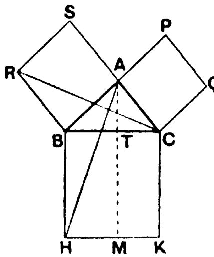


FIG. 291.

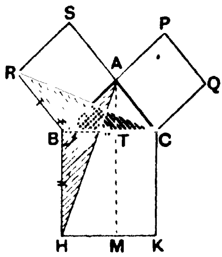


FIG. 292.

Let ABC be a triangle having a rt. ∠ at A.
Then we have to prove that

$$BC^2 = CA^2 + AB^2.$$

Construction. On the sides describe sqs. as in the figure. Draw ATM ∥ BH or CK. Join RC, AH.

Proof. Since the ∠ˢ BAS, BAC are rt. ∠ˢ;

∴ SAC is a str. line.

In the two △ˢ ABH and RBC,

(i) AB = RB, (sides of a sq.)

(ii) BH = BC, (sides of a sq.)

(iii) inc. $\hat{ABH} = 90° + \hat{ABC}$ = inc. $\hat{RBC}$;

∴ △ABH ≡ △RBC. (2 sides, incl. angle)

But △ABH = ½BHMT (same base BH; same ∥ AM, BH)
and △RBC = ½ABRS, (same base RB; same ∥ RB, SC);

∴ BHMT = ABRS.

Similarly, by joining AK and BQ, we can prove

$$TMKC = ACQP;$$

$$\therefore \text{ by addition, } BCKH = ABRS + ACQP,$$

$$i.e. \text{ sq. on } BC = \text{sq. on } AB + \text{sq. on } AC.$$

Note the first proof of this Theorem is ascribed to Pythagoras (about 500 B.C.), though the above proof is due to Euclid (about 300 B.C.). The Egyptians seem to have been acquainted with some applications of the theorem.

Perigal's method by Dissection.

Through the middle point of the sq. on AB, draw lines $\parallel$ and $\perp^{ar}$ to BC.

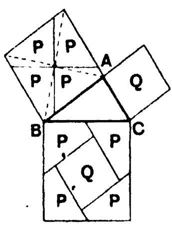


FIG. 293.

Through the middle points of the sides of the square on BC draw lines $\parallel$ to AB and AC alternately.

The areas marked P are all congruent, and those marked Q are equal squares

Thus the squares on AB and AC can be made to exactly fill up the sq. on BC.

Arithmetical Notes.

(i) If two sides of a right-angled triangle are given, it is easy to calculate the third ; *e.g.* if the sides containing the right angle are 3 and 4 ins., the length of the hypotenuse is given by the equation

$$h^2 = 3^2 + 4^2 = 9 + 16 = 25 ;$$

$$\therefore\ h = \sqrt{25} = 5 \text{ ins.}$$

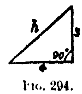


FIG. 294.

This fact, that if the sides of a $\triangle$ are 3, 4, 5 units, the angle opposite the greater side is a right angle, is one of the practical methods of laying out two lines at right angles on the ground.

Also, if the hypotenuse and one side are 7 and 3 ins., the third side x is given by the equation

$$7^2 = 3^2 + x^2 ;$$

$$\therefore\ x^2 = 49 - 9 = 40 ;$$

$$\therefore\ x = \sqrt{40} = 6{\cdot}32 \text{ ins.}$$

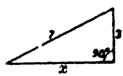


FIG. 295

(ii) This suggests an easy method for finding the square root of 40 by accurate drawing, and also of any other number which can be expressed as the sum or difference of two square numbers.

(iii) In the case of a right-angled $\triangle$ ABC (Fig. 296) in which

$$\hat{BAC} = \hat{ABC} = 45°,$$

let AC = CB = 1, then

$$AB^2 = AC^2 + CB^2 = 2 ;$$

$$\therefore\ AB = \sqrt{2}.$$

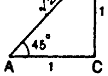


FIG. 296.

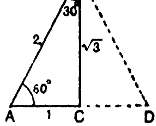


FIG. 297.

(iv) In an equilateral $\triangle$ABD (Fig. 297), if the angle ABD is bisected by BC, then $\hat{ABC} = 30°$ and $\hat{BAC} = 60°$.

Since the $\triangle^s$ ABC, DBC are congruent,

$$AC = \tfrac{1}{2}AD = \tfrac{1}{2}AB\ ;$$

$$\therefore\ \text{since } AB^2 = AC^2 + CB^2,$$

$$4 = 1 + CB^2\ ;$$

$$\therefore\ CB^2 = 3,\quad i.e.\ CB = \sqrt{3}.$$

To find the area of a triangle ABC, if AB $=5$ cms., BC $=7$ cms., CA $=6$ cms.

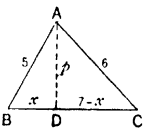


FIG. 298

Draw a perp. AD from A to BC, let BD $=x$ cms., AD $=p$ cms. From the right-angled $\triangle$ ABD,

$$p^2 + x^2 = 25\ ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

from the $\triangle$ ADC, $\quad p^2 + (7-x)^2 = 36\ ;$

$$\therefore\ 25 - x^2 = 36 - (7-x)^2, \text{ by equating values of } p^2\ ;$$

$$\therefore\ 14x = 38,$$

$$i.e.\ x = \tfrac{19}{7}.$$

Substituting in (i),

$$p^2 = 25 - \left(\tfrac{19}{7}\right)^2 = \tfrac{864}{49}\ ;$$

$$\therefore\ p = AD = \frac{\sqrt{864}}{7}.$$

$$\text{Area of } \triangle ABC = \tfrac{1}{2}BC \,.\, AD = \tfrac{1}{2} \,.\, 7 \,.\, \frac{\sqrt{864}}{7} = \sqrt{216}$$

$$= 14{\cdot}7 \text{ sq. cms.}$$

THEOREM 23.

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the angle contained by these two sides is a right angle.

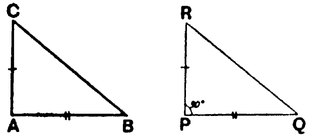


FIG. 299

Let ABC be a in which

$$AB^2 + CA^2 = BC^2.$$

Then we have to prove that $\hat{A}$ is a rt. ∠.

Construction. Describe another triangle having $\hat{P} = 90°$ and $PQ = AB$, $PR = AC$.

Proof. $QR^2 = PQ^2 + RP^2$ (∵ $\hat{P} = 90°$)

$= AB^2 + CA^2$ (Construction)

$= BC^2$; (Hypothesis)

∴ $QR = BC$.

∴ $\triangle ABC \equiv \triangle PQR$, (3 sides)

and in particular $\hat{A} = \hat{P}$.

But $\hat{P}$ was drawn a rt. ∠,

∴ $\hat{A} = 90°$.

EXERCISE 27.

1. Calculate the lengths of the diagonals of rectangles in which the sides are

(i) 5 cms., 6 cms.,

(ii) 3 cms., 4 cms.,

(iii) 4·2 ft., 5·7 ft.

2. Calculate the third side of a right-angled $\triangle$ABC, given that angle A $= 90°$, and

(i) $a = 10$ cms., $b = 8$ cms.,

(ii) $a = 8{\cdot}4$ cms., $c = 4{\cdot}3$ cms.,

(iii) $b = 4{\cdot}9$ cms., $c = 6{\cdot}7$ cms.

3. Of the three sides of a right-angled triangle the greatest is 8 inches and the least 5 inches long; find the length of the other side correct to two significant figures.

4. A telegraph pole is steadied by a tight wire rope fastened to the pole at a point 16 feet above the ground and to a peg in the ground 9 feet from the foot of the pole; calculate, within half an inch, the length of the wire if the pole is upright.

5. Construct an accurate figure from which may be read the values of $\sqrt{10}$, $\sqrt{91}$, $\sqrt{28}$, $\sqrt{37}$, and state what they are.

6. Construct a right-angled triangle with sides 5 and 6, and thus find $\sqrt{61}$.

7. Construct a right-angled triangle with sides $\sqrt{5}$ and $\sqrt{6}$, and thus find $\sqrt{11}$.

8. ABCD is a square, side 2 ins. The diagonal AC is produced to K, so that CK $= \frac{1}{2}$AC; KM is drawn $\perp^{ar}$ to AB produced; find the length of KM by accurate drawing, and verify by calculation.

9. If the length of a lawn-tennis court is 78 ft., and the breadth 36 ft., calculate to $\frac{1}{10}$ ft. the distance between two opposite corners.

10. A room is 18 ft. long, 13 ft. broad and 11 ft. high. Calculate (to the nearest inch), the diagonals of (i) a long wall, (ii) a short wall, (iii) the ceiling.

11. A man walks 4 miles East, then 3 miles North, and afterwards 2 miles East. Represent this on squared paper and obtain the

distance between his original and final positions by (i) measurement, (ii) calculation.

12. Repeat Qn. 11, if the man walks 5 miles East, 3 miles North, 2 miles West.

13. The base of an isosceles triangle is 14 inches and the equal sides are each 25 inches. Calculate the height of the triangle and thence its area.

14. Find the area of an equilateral $\triangle$ if the side is 11 inches.

15. A wire is stretched between the tops of two posts, which are 22 ft. and 13 ft. high, and 12 ft. apart. Calculate the length of the wire (Fig. 300).

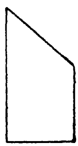


FIG. 300.

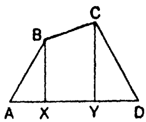


FIG. 301.

16. A wire ABCD (Fig. 301) is stretched from A to D over the tops of two posts BX, CY.

If AX=10 ft., XY=13 ft., YD=12 ft., BX=15 ft., CY=19 ft., calculate the length of the wire.

17. A section of a railway cutting is in the shape of a trapezium ABCD. If BC=26 ft., AD=38 ft., and the depth of the cutting 8 ft., calculate AB, if AB=DC (Fig. 302).

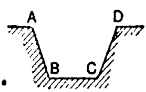


FIG. 302.

18. If the diagonals of a rhombus are 6 and 11 cms. respectively, calculate the lengths of the sides and the area of the rhombus.

(Note that the diagonals bisect one another at right angles.)

19. If the sides of a rhombus are each 5·7 cms. and a diagonal 8·6 cms., calculate the length of the other diagonal.

20. The edge of a cube is 2·3 inches. Draw one face of the cube and measure the diagonal of the face in inches. Draw also a section made by a plane drawn through a pair of opposite edges of the cube. Measure the diagonal of the cube in inches.

21. The height of a cone is 3·1 ins., and the diameter of the base is 2·6 ins. Find the distance from the vertex to any point of the rim of the base.

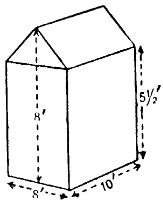


FIG. 303

22. A hut has the dimensions shown in the figure. Find (i) the area of the four walls, (ii) the area of the roof.

23. A pyramid has a square base, of side 3 inches, and each of its other edges is of length 4 inches. Find the perp. distance (OM) of the vertex from the base, and the total surface area of the pyramid.

(Find BM and note that angle OMB = 90°.)

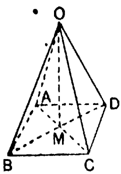


FIG. 304.

24. A pyramid OABCD has a square base ABCD. Each side of the square ABCD is 8 feet, and each of the slant edges OA, OB, OC, OD is 9 feet. Find the height of O above the base.

25. A room is 20 ft. long, 16 ft. wide, and 10 ft. high; a cord is stretched straight from one corner of the floor to the point where the diagonals of the ceiling intersect. Estimate from drawings the length of this cord and the distance of its middle point from the other corner of the same end of the floor (Fig. 305).

(Find AE by drawing and construct rect. ACDE; mark M and draw AOB.)

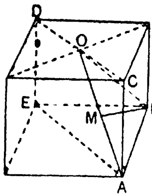


Fig. 305.

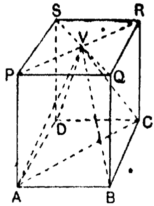


Fig. 306.

26. Fig. 306 is a rough sketch of a rectangular block, the ends ABCD and PQRS being equal squares of side 3 inches. The edges PA, QB, RC, SD are each 3·6 inches. A pyramid VABCD is inscribed in the block, having its vertex V at the intersection of the diagonals of the square PQRS.

Draw full size the section of the figure by the plane PRCA, and draw also the true shape of one of the sloping faces (VAB) of the pyramid. Calculate the length of one of the sloping edges (VA) of the pyramid and check by measurement from your figure.

27. A room is 16 ft long, 12 ft broad and 10 ft high. A fly is in the centre of an end wall and wishes to go to a point on the opposite wall situated 2 ft from the side and 2 ft from the ceiling. Calculate the shortest distance it will have to travel, (i) by walking round the walls, (ii) by crossing the ceiling, (iii) by flying.

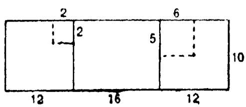


Fig. 307.

(In the first case unfold the walls, so that a long wall and the two end walls are all in one plane.)

28. Prove that the square on the diagonal of a given square is twice as large as the original square.

29. In any $\triangle ABC$, AD is drawn perpendicular to BC; prove that

$$AC^2 \sim AB^2 = DC^2 \sim DB^2.$$

30. K is any point within a rectangle ABCD; prove that

$$AK^2 + CK^2 = BK^2 + DK^2.$$

31. ABC is a triangle right-angled at B, and X any point in BC; prove that
$$AX^2+BC^2=AC^2+BX^2.$$

32. ABCD is a quadrilateral such that the diagonal AC is perp. to the diagonal BD. Prove that
$$AB^2+CD^2=AD^2+BC^2.$$

33. In an equilateral $\triangle$ABC, if AX is perp. to BC, prove that
$$AX^2=\tfrac{3}{4}BC^2.$$

34. In a rhombus the lengths of the sides are each a, and the diagonals x and y respectively; prove that
$$4a^2=x^2+y^2.$$

35. ABC is a triangle, PD, PE, PF are perpendiculars from a point inside to BC, CA, AB respectively; prove that
$$AF^2+BD^2+CE^2=AE^2+BF^2+CD^2.$$

36. In a right-angled triangle ABC, right-angled at C, if P and Q are the middle points of BC, CA, respectively; prove that
$$4(AP^2+BQ^2)=5AB^2.$$

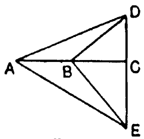


FIG. 308.

37. If AC is perp. to DE (Fig. 308), prove that
$$AD^2-BD^2=AE^2-BE^2.$$

38. ABC is an equilateral triangle, and $CY=\tfrac{1}{4}CB$; prove that
$$AY^2=13YC^2.$$

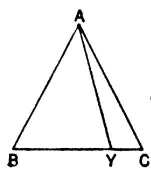


FIG. 309.

39. P and Q are any two points in the sides AC, AB of a triangle, right-angled at A. Prove that
$$PQ^2+BC^2=BP^2+CQ^2.$$

40. ABCD is a quadrilateral such that the sides DA and CB meet at right angles when produced; prove that

$$AC^2+BD^2=AB^2+CD^2.$$

41. ABCD is a square; points E, F, G, H are taken on AB, BC, CD, DA such that AE=BF=CG=DH=$\frac{1}{4}$ of the side of the square. Show that the area of the square EFGH is $\frac{5}{8}$ of the area of the square ABCD.

42. ABCD is a square; AB is produced to P, so that BP=2AB; and BC, CD, DA are similarly produced to Q, R, S; prove that PQRS is a square having 13 times the area of the original square.

43. ABCD is a square whose side is 10 cms.; PQR is an equilateral triangle having P at the middle point of AB, Q on AD and R on BC. Calculate the lengths of AQ and AR.

PART II.

CIRCLES.

THEOREM 24.

The straight line drawn from the centre of a circle to bisect a chord is at right angles to the chord, and conversely, the straight line drawn from the centre of a circle at right angles to a chord bisects the chord.

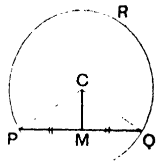


FIG. 310

(i) Let PQ be any chord of a circle PQR, C the centre and M the middle point of PQ.

We have to prove CM is $\perp^r$ to PQ.

Construction. Join CP, CQ, CM.

Proof. In the $\triangle$s CPM, CQM,

(i) CP = CQ, (radii of same $\odot$)

(ii) PM = MQ, (Hypoth.)

(iii) CM is common,

$\therefore$ $\triangle$CPM $\equiv$ $\triangle$CQM, (3 sides)

and in particular, $C\hat{M}P = C\hat{M}Q = 90°$; (adj angles)

$\therefore$ CM is perp. to PQ.

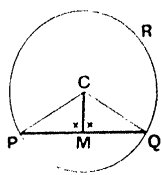


FIG. 311.

(ii) Let PQ be any chord of a circle PQR, C the centre and CM $\perp^r$ to PQ.

We have to prove PM = MQ.

Construction. Join CP, CQ.

Proof. In the right-angled $\triangle$s CPM, CQM,

(i) CP = CQ, (radii of same circle)

(ii) CM is common;

$\therefore$ $\angle$CPM $\equiv$ $\angle$CQM, (hyp. and 1 side)

and in particular, PM = MQ;

$\therefore$ CM bisects PQ.

It follows from this proposition that

the perpendicular bisector of a chord of a circle must pass through the centre of the circle.

(i) For the $\perp^{\text{ar}}$ bisector is the locus of all points equidistant from the extremities of the chord.

Now, by Definition, the centre of the circle is a point equidistant from the extremities of the chord.

Therefore the $\perp^{\text{ar}}$ bisector must go through the centre of the circle.

(ii) Also, if XY is the diameter produced and A, B, two corresponding points, so that B is the image of A, XY being

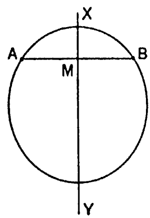


FIG. 312

an axis of symmetry, then it follows that AM = MB, and angles AMX, BMX are right angles.

THEOREM 25.

There is one circle and only one circle which passes through three given points not in the same straight line.

Let A, B, C, be the three given points.

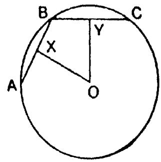


FIG. 313

Construction. Join AB, BC, and bisect them at right angles by the lines XO, YO respectively.

Proof. AB is a chord of any ⊙ passing through A and B;

∴ the centre of the ⊙ lies on XO.

BC is a chord of any circle passing through B and C;

∴ the centre of the ⊙ lies on YO.

If O is the point of intersection of XO and YO, it follows that OA = OB = OC, and that a circle with centre O will pass through A, B and C.

Since XO, YO meet in only one point, there can be only one circle.

COR. Two circles cannot intersect in more than two points.

Note that if A, B, C are in one straight line, then XO, YO are parallel and do not meet, so that the circle cannot be drawn.

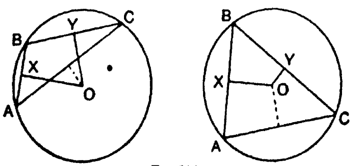


FIG. 314.

Also, if AC is joined, the circle *circumscribes* the △ABC, and since the perp. bisector of the chord AC must pass

through the centre of the circle, it follows that *the three perp. bisectors of the sides of a triangle are concurrent*, *i.e.* pass through the same point.

Another proof will now be given of this important fact.

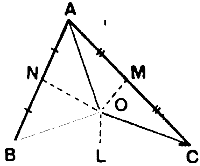


FIG. 315.

Let ABC be a triangle, OM and ON the perpendicular bisectors of two sides. L the middle point of BC. Join LO.

We have to prove that LO is the perpendicular bisector of BC.

Proof. In the $\triangle$s OMA, OMC,

(i) MA = MC, (Hypoth.)

(ii) $\hat{OMA} = \hat{OMC}$, (rt. angles)

(iii) OM is common ;

$\therefore$ $\triangle$OMA ≡ $\triangle$OMC, (2 sides, incl. ang.)

and in particular, OA = OC.

Similarly, $\triangle$ONA ≡ ONB,

and OA = OB.

It follows that OB = OC.

In the $\triangle$s OLB, OLC,

(i) OB = OC, (just proved)

(ii) BL = LC, (construct.)

(iii) OL is common ;

$\therefore$ $\triangle$OLB ≡ $\triangle$OLC, (3 sides)

and in particular $\hat{OLB} = \hat{OLC}$,

and since these are adjacent angles,

$\therefore$ OL is perp. to BC,

i.e. the three perp. bisectors of the sides of the triangles are concurrent.

N.B.—A circle, centre O and radius OA or OB or OC, passes through each angular point of the triangle. The circle is called the **circumcircle** and O is called the **circumcentre.**

THEOREM 26.

Equal chords in a circle must be equidistant from the centre. And conversely, chords which are equidistant from the centre must be equal. (See Ex. 28. 27.)

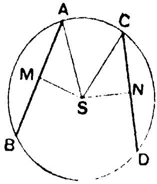


FIG. 316.

Let AB, CD be two chords of a circle, centre S.

Construction. Join SA, SC, and draw SM, SN $\perp^{\text{ar}}$ to AB, CD.

(i) *Proof.* **Given** AB = CD **we have to prove** SM = SN.

M and N are the middle points of AB and CD;

$\therefore$ AM = $\frac{1}{2}$AB = $\frac{1}{2}$CD = CN.

In the right-angled $\triangle^s$ AMS, CNS,

(i) AS = CS, (radii of same circle)

(ii) AM = CN; (just proved)

$\therefore$ $\triangle$AMS $\equiv$ $\triangle$CNS, (hyp. and 1 side)

and in particular, SM = SN.

(ii) Conversely,

Given SM = SN, **we have to prove** AB = CD.

In the right-angled $\triangle^s$ AMS, CNS,

(i) AS = CS, (radii of same circle)

(ii) SM = SN; (Hypoth.)

$\therefore$ $\triangle$AMS $\equiv$ $\triangle$CNS,

and in particular, AM = CN.

M, N are the middle points of AB, CD;

$\therefore$ AB = 2AM = 2CN = CD.

EXERCISE 28.

1. Describe a circle with radius 5 cms. and passing through two points 6 cms. apart. Measure and calculate the distance of the chord from the centre of the circle.

2. Draw a circle, radius 4·4 cms., and place in it two chords each of length 5 cms. Measure and calculate the distance of each chord from the centre.

3. Draw a circle, radius 3·8 cms, and through a point 2·2 cms. from the centre draw a chord having this point as its middle point. Measure and calculate the length of this chord. Can you draw a shorter chord through this point ?

4. Draw a circle, radius 5 cms. ; place in it 3 chords of lengths 4, 5, 6 cms., and measure and calculate the distance of each chord from the centre.

5. A stream 20 feet wide is to be spanned by a bridge with a circular arch, 7 feet high at the middle. Find, by construction, the radius of the arch and the angle it subtends at the centre of the circle.

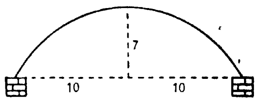


FIG. 317.

6. Draw a circle, radius 4 cms., and take a point 3 cms. from the centre. Draw two chords through this point inclined at 36° to the diameter through the point. Measure the distances of the chords from the centre

7. Draw two circles, radii 3·2 and 4·8 cms, having their centres 5·6 cms. apart. Measure the two parts into which the line of centres divides the common chord (See Qn. 18.)

8. Draw a triangle B=72°, C=58°, c=4·2 cms., and describe a circle passing through A, B and having its centre on BC. Measure the radius.

9. If the lengths of two parallel chords of a circle (on the same side of the centre), are 6 and 8 cms., and their distance apart 2 cms., find the radius of the circle by construction.

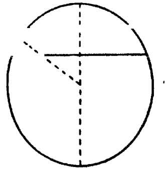


FIG. 318.

10. Given two points A and B which are 6 cms. apart, draw a number of circles passing through A and B. What is the locus of their centres ?

11. If $AB=4\cdot2$ cms., $BC=3\cdot4$ cms., and $\hat{ACX}=42°$, draw a circle to pass through A and B and with its centre on CX. Measure its radius.

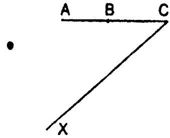


FIG. 319

For what value of the angle ACX would it be impossible to find the centre ?

12. Draw a triangle with two sides measuring 3·6 and 4 inches and including an angle of 52°.

Describe a circle about the triangle and measure its radius.

13. Draw a triangle ABC in which

$AB=6$ cms., $BC=7$cms.,

and the angle $A=54°$.

Construct the circumcircle of the triangle, and measure its diameter.

14. The sound of the discharge of an enemy long-range gun is heard simultaneously at three listening stations A, B, C (Fig. 320). Prick the figure through on to your paper. Assuming that the sound travels with equal velocity in all directions, mark the location of the gun, and find its distance (in yards to the nearest 100 yards) and bearing from the battery at P. (*Army.*)

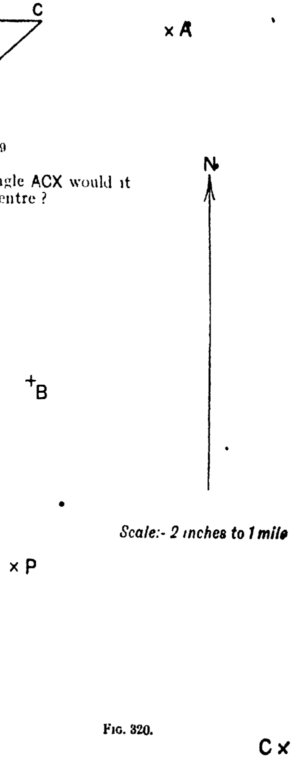


FIG. 320.

15. If two parallel chords of a circle are 6·4 and 8·4 cms. long, and the radius of the circle is 5 cms., calculate the distance between the chords, if they are on

(i) the same side of the centre,
(ii) opposite sides of the centre.

16. If two parallel chords on the same side of the centre are 2·4 cms. apart, and their lengths are 7·2 and 9·6 cms., calculate the radius of the circle.

17. Prove that the locus of the middle points of a system of parallel chords of a circle is a diameter.

18. Prove that the common chord of two circles which intersect is bisected at right angles by the line joining the centres.

(Join X, the middle pt. of AB to C and D. Also join CA, CB, DA, DB.)

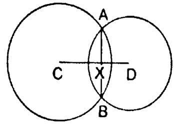


FIG. 321.

19. Two circles intersect and a line is drawn through either point of intersection parallel to the line of centres and terminated by the circumferences. Prove that it is double the length of the line joining the centres.

20. ABC is an equilateral triangle, and a circle with A as centre cuts BC at D and E; prove BD = CE.

21. A diameter AB of a circle bisects a chord CD; if BC is parallel to AD, prove that CD is also a diameter.

22. AB, AC are two equal chords of a circle, centre S; SM, SN are perpendiculars to the chords; prove that ∠AMS △ANS.

23. P is any given point on a given circle whose centre is O. From P a fixed length PQ is drawn in a fixed direction. Find the locus of Q.

24. A straight line cuts two concentric circles in A, B, C, D. Prove that AB = CD (Fig. 322).

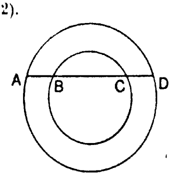


FIG. 322.

25. AB is a diameter of a circle, centre S, AL, SM, BN are perpendiculars to a chord CD which does not intersect AB; prove M bisects LN, and CL = DN.

26. AB, CD are equal chords of a circle ABCD; P is a point in AB and Q a point in CD such that AP = CQ; prove that P and Q are equidistant from the centre of the circle.

27. In any circle, prove that the longer of two chords must be nearer the centre than the shorter, and conversely, the chord which

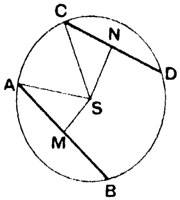


Fig. 323.

is nearer the centre must be longer than one more remote (Use Pythagoras' Theorem for the △s SAM, SCN.)

An **Arc** of a circle is the portion of the circumference between any two points A, C. (Curved line ABC in Fig. 324.)

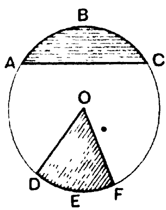


Fig. 324.

A **Segment** of a circle is the area contained between a chord and the arc which it cuts off. (Area ABC.)

A **Sector** of a circle is the area bounded by two radii and the part of the circumference which they intercept. (Area ODEF.)

ANGLES AND ARCS.

THEOREM 27.

The angle which an arc of a circle subtends at the centre is double of that which it subtends at any point on the remaining part of the circumference.

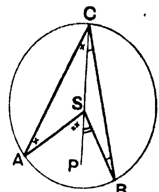


FIG. 325.

Let ABC be any circle, centre S.

We have to prove that $A\hat{S}B = 2A\hat{C}B$.

Construction. Join CS and produce to any point P.

Proof. In $\triangle ASC$, $AS = SC$; (radii of same circle)

$\therefore\ S\hat{A}C = S\hat{C}A$; (Isos. Δ)

$\therefore$ ext. $A\hat{S}P = S\hat{A}C + S\hat{C}A = 2S\hat{C}A$.

Similarly $B\hat{S}P = 2S\hat{C}B$.

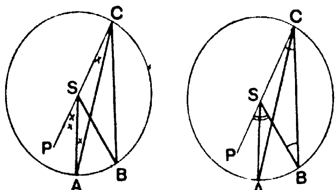


FIG. 326.

Hence, Fig. 325, $A\hat{S}B = A\hat{S}P + B\hat{S}P$

$= 2S\hat{C}A + 2S\hat{C}B$

$= 2A\hat{C}B$;

and Fig. 326, $A\hat{S}B = B\hat{S}P - A\hat{S}P$

$= 2B\hat{C}S - 2A\hat{C}S$

$= 2A\hat{C}B$.

THEOREM 28.

Angles in the same segment of a circle are equal to one another.

Let APB, AQB be two angles in the same segment APQB of a circle centre O.

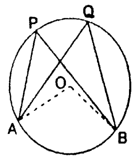


FIG. 327.

Construction. Join O to A and B.

Proof. $A\hat{O}B = 2A\hat{P}B$,

since the angle at the centre is double the angle at circumference;

$$\text{similarly } A\hat{O}B = 2A\hat{Q}B;$$

$$\therefore A\hat{P}B = A\hat{Q}B.$$

Four points which lie on the circumference of a circle are said to be *concyclic*.

THEOREM 29.

If the straight line joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic.

If the straight line AB subtends equal angles APB, AQB at P and Q, we have to prove that A, P, Q, B are concyclic.

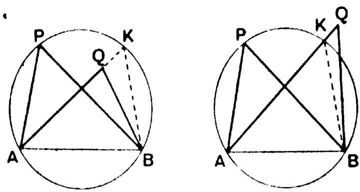


FIG. 328

Construction. Draw the circle BAP; if it does not pass through Q it will meet AQ or AQ produced at some point K.

Join KB.

Proof. $\widehat{APB} = \widehat{AKB}$, (angles in same segment)

but $\widehat{APB} = \widehat{AQB}$; (hypoth.)

$\therefore \widehat{AKB} = \widehat{AQB}$;

i.e. the exterior angle of a $\triangle$ equals the interior opposite angle, and this is impossible;

hence the circle through A, B, P must also pass through Q.

Note. The *locus* of a point at which the straight line joining two given points subtends a constant angle is an arc of a circle passing through the two given points.

CONSTRUCTION 7.

On a given straight line to describe a segment of a circle containing an angle equal to a given angle.

(i) On AB describe a segment of a circle containing an *acute* angle of 50°.

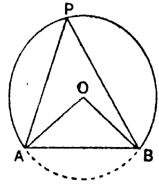


FIG. 329.

Construction. Draw two lines AO, BO, so that

$$O\hat{A}B = O\hat{B}A = 90° - 50° = 40°.$$

With centre O and radius OA construct a segment of a circle.

If P is any point in the arc, we have to prove that $A\hat{P}B = 50°$.

Proof. $$A\hat{O}B = 180° - (O\hat{A}B + O\hat{B}A)$$
$$= 180° - 80° = 100°,$$

$$\therefore\ A\hat{P}B = \tfrac{1}{2}A\hat{O}B = 50°.$$

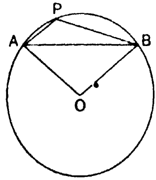


FIG. 330

(ii) On AB describe a segment of a circle containing an *obtuse* angle of 130°.

Construction. Draw two lines AO, BO, so that

$$O\hat{A}B = O\hat{B}A = 130° - 90° = 40°.$$

With centre O and radius OA construct a segment of a circle.

If P is any point on the arc, we have to prove that $A\hat{P}B = 130°$.

Proof.

$$\hat{AOB} = 180° - (\hat{OAB} + \hat{OBA})$$
$$= 180° - 80° = 100°;$$
$$\therefore \hat{APB} = \tfrac{1}{2} \text{ reflex } \hat{AOB}$$
$$= \tfrac{1}{2}(360° - \hat{AOB})$$
$$= \tfrac{1}{2}(360° - 100°)$$
$$= 130°.$$

Note that in each case the angles OAB, OBA are made equal to the difference of the given angle and 90°.

It should be noticed that this construction supplies the

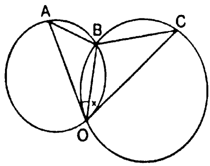


FIG. 331.

method, used practically, for finding an observer's position (O) on a map, if three objects A, B, C, which are observed in the landscape, can be identified on the map.

The observer can measure the angles AOB, BOC by means of a sextant, and then on the map construct segments of circles on AB, BC containing angles equal to the known angles AOB, BOC.

The second point of intersection of these circles determines the position of the observer O on the map.

EXERCISE 29.

1. ASB is a diameter of a circle, centre S, radius 2·4 cms. AP and AQ are chords making $P\hat{A}S = 70°$, $Q\hat{A}S = 32°$. Measure $P\hat{A}Q$, $P\hat{S}Q$ and $P\dot{Q}$. What is the relation between the two angles measured?

2. ASB is a diameter of a circle, centre S, radius 2·6. AP_1 is a chord bisected at M_1 and making $P_1AB = 20°$. N is the mid-point of AS. Measure NM_1. AM_2P_2, AM_3P_3, AM_4P_4 are similar chords, making $P_2AB = 40°$, $P_3AB = 60°$, $P_4AB = 80°$. Measure NM_2, NM_3, NM_4. Explain your results.

3. ABC is a $\triangle$ inscribed in a circle, centre O, and $O\hat{A}C = 25°$, $O\hat{B}C = 30°$. Find the magnitudes of the angles ACB, AOB (Fig. 332).

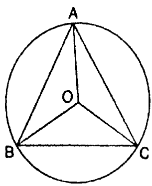


FIG. 332.

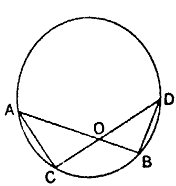


FIG. 333.

4. If AB and CD are two chords of a circle intersecting at O, and $A\hat{C}D = 100°$, $A\hat{O}D = 140°$, what are the magnitudes of the angles CAO, OBD? (Fig. 333).

5. Slide a set square between two drawing pins A and B, and mark the positions of the angular point P for various different positions of the set square; and thus trace out the locus of P (Fig. 334).

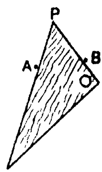


FIG. 334.

6. On a base 2·3 inches long, describe a segment of a circle containing an angle of 60°. Measure the radius of the circle in inches

7. On a base of 6·7 cms, describe a segment of a circle containing an angle of 75°. Measure the height of the segment.

8. On a base of 7·8 cms., describe a segment of a circle containing an angle of 110°. Measure the radius of the circle.

9. On a base of 3·2 inches, describe a segment of a circle containing an angle of 136°. Measure the radius of the circle.

10. Construct a triangle on a base of 4 inches, altitude 2 inches, and vertical angle 70°. Measure the perimeter.

11. OA, OB are two straight lines inclined at 40°, and X, Y are points on OA, such that OX = 3 cms., and OY = 8 cms. Find the points in OB at which XY subtends an angle 55°. Measure their distances from O.

12. Upon a base of 7 cms. construct a triangle with vertical angle 40°, and area 17·5 square centimetres. Measure the two angles at the base.

13. Construct on a base 3·7 inches long a triangle whose vertical angle is 112° and whose area is the greatest possible. Measure its sides

14. A, B, C are three points such that AB is 5 miles, BC is 10 miles, and CA is 7 miles P is a point in the plane of ABC such that $B\hat{P}A$ is 30°, $C\hat{P}A$ is 30° and $B\hat{P}C$ is 60°. Find, by measurement, the distances of P from A, B, and C

15. From two stations A, B, 350 yards apart, two trees C, D, on the same side of AB, are observed. By measurement it is found that ∠BAD = 95°, ∠BAC = 44°, ∠ABD = 27°, and ∠ABC = 78°. Draw a plan of the relative positions of A, B, C and D, taking 1 inch to represent 100 yards, and from it estimate the distance between the trees

Prove that a circle can be drawn to pass through A, B, C, D.

16. A, B, C are three stations on shore, all of which can be seen from a boat P. AB = 35 chains, BC = 29·3 chains B is due east of A, and the bearing of BC is 37° east of south The boat is at anchor on the south side of AB. To locate the position of the boat, observations of the angles APB and BPC are taken with a sextant, and these give APB = 60°, and BPC = 76°.

Make a drawing to scale, and determine by geometrical construction the position of the boat From measurement of your figure give the distance BP.

17. A ship is torpedoed and sunk while steaming near a straight coast line on which there are two lighthouses, A and B, 10 miles apart. There are two survivors, the first states that the ship was between 4 and 6 miles from the coast; the second, that the angle subtended at the ship by the line AB was between 60° and 90°.

Draw a plan, on the scale 1 cm. to 2 miles, and shade the area within which the wreck must lie. (*Army*.)

18. Draw any quadrilateral ABCD in a circle; join AC and BD, and prove that $\hat{D}$ = the sum of two of the angles of the △ABC. Hence show that $\hat{B}$, $\hat{D}$ are supplementary.

19. AB and CD are parallel chords of a circle with centre O. If AD and BC intersect at X, prove that A, C, O, X are concyclic.

20. O is the point of intersection of the perpendiculars from the angular points of a triangle to the opposite sides, and the altitude AOD produced meets the circumcircle in P. Prove that both the angles OBC and PBC are complements of C. Hence prove that OP is bisected at D.

21. ABC is an acute-angled triangle, and O is the centre of the circle which passes through A, B, C. Show that the angles OBC and BAC are complementary.

22. ABCD is a cyclic quadrilateral and AD > CD; BE is drawn meeting AC at E and making $\hat{ABE} = \hat{DBC}$. Prove that ABE and BCD, ABD and BEC are pairs of equiangular triangles (Fig. 335).

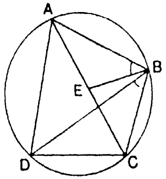


FIG. 335

23. ABC is an equilateral triangle inscribed in a circle, and M is any point on the smaller arc BC. The lines AM, BM, CM are drawn, and AP is taken along AM and equal to BM. Prove that the triangles APC, BMC are congruent.

24. Two circles, whose centres are A and B, intersect in C; AC produced cuts the circle whose centre is B in H, and BC produced cuts the circle whose centre is A in K. Prove that A, B, H, K are concyclic.

25. From a point O, external to a circle, two straight lines, OAB, OCD are drawn, cutting the circle at A, B and at C, D. Prove that the angle OAD is equal to the angle OCB.

26. Two circles PAB, QAB, whose centres are H and K, intersect in A and B; and any line PAQ through A cuts the circles in P and Q. Prove that the triangles BPQ, AHK are equiangular.

27. Perpendiculars from A and B on the opposite sides of the acute-angled triangle ABC cut in H. Show that the angle AHB is the supplement of the angle C.

Deduce that if AH cuts the circle circumscribing ABC again in J, then the triangle BHJ is isosceles.

28. AB, CD are two parallel chords of a circle. Prove that, if AD, BC meet at T, and AC, BD meet at S, TS passes through the centre of the circle.

29. Given the base and vertical angle of a triangle, find the locus of its centroid.

(If BC is the given base, A the vertex and G the centroid, draw GM, GN parallel to AB, AC respectively, and show that M and N are fixed points.)

THEOREM 30.

The angle in a semicircle is a right angle.

If AOB is the diameter of a circle with centre O, and APB an angle in the semicircle, we have to prove that $\hat{APB}$ is a right angle.

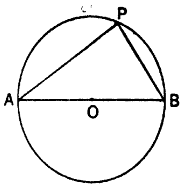


FIG. 336.

Proof. Since the angle at the centre is twice the angle at the circumference,

$$\therefore\ \hat{AOB} = 2\hat{APB}.$$

But $$\hat{AOB} = 2 \text{ right angles};$$

$$\therefore\ \hat{APB} = 1 \text{ right angle}.$$

THEOREM 31.

The circle described on the hypotenuse of a right-angled triangle as diameter passes through the opposite vertex.

Let ABC be a triangle right-angled at C; it is required to prove that a circle on AB as diameter passes through C.

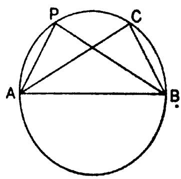


FIG. 337.

Construction. Describe a circle on AB as diameter, and in the semicircle above AB take any point P. Join PA, PB.

Proof. Since the angle APB is in a semicircle;

$$\therefore\ \hat{APB} = 1 \text{ right angle};$$

$$\text{but } \hat{ACB} = 1 \text{ right angle};$$

$$\therefore\ \hat{APB} = \hat{ACB}.$$

Thus, since AB subtends equal angles at P and C, it follows that a circle passes through A, P, C, B.

i.e. C is on the circle with AB for diameter.

A **cyclic quadrilateral** is a quadrilateral through the four angular points of which it is possible to draw a circle.

CYCLIC QUADRILATERAL.

THEOREM 32.

The opposite angles of a cyclic quadrilateral must be supplementary.

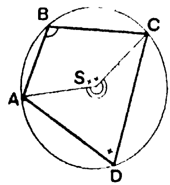


FIG. 338.

Let ABCD be a cyclic quadrilateral.

We have to prove $\hat{A}+\hat{C}=\hat{B}+\hat{D}=180^\circ$.

Construction. Join A and C to S, the centre of the circle.

Proof. Since the angle at the centre is double the angle at the circumference,

$$\therefore\ \widehat{ASC}=2\hat{D},$$

the reflex angle ASC $=2\hat{B}$;

$\therefore$ the total angle at S $=2\hat{B}+2\hat{D}$;

but the total angle at S $=4$ right angles;

$$\therefore\ 2\hat{B}+2\hat{D}=360^\circ;$$

$$\therefore\ \hat{B}+\hat{D}=180^\circ;$$

and since $\hat{A}+\hat{B}+\hat{C}+\hat{D}=360^\circ$,

$$\therefore\ \hat{A}+\hat{C}=180^\circ.$$

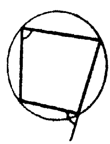


FIG. 339.

COR. An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

THEOREM 33.

If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.

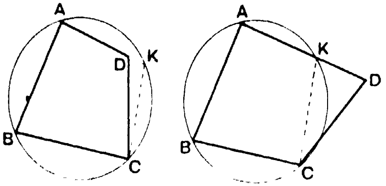


FIG. 340.

Let ABCD be a quadrilateral having $\hat{B} + \hat{D} = 180°$.
We have to prove that ABCD is cyclic.

Construction. Draw the circle ABC and if it does not pass through D, let it meet AD or AD produced at K.

Proof. $\hat{B} + \hat{K} = 180°$, (ABCK cyclic quad.)

but $\hat{B} + \hat{D} = 180°$; (Hypoth.)

$\therefore \hat{D} = A\hat{K}C$,

which is impossible; (Th. 5)

$\therefore$ the circle ABC must pass through D.

COR. If an exterior angle of a quadrilateral is equal to the interior and opposite angle, the quadrilateral must be cyclic.

CONSTRUCTION 8.

To cut off from a given circle a segment containing an angle equal to a given angle.

Let A be the given angle.

Draw PQ any chord of the circle.

Make $\hat{QPR} = \hat{A}$.

(i) Join QR. Then RQ cuts off the segment RPQ which contains the required angle.

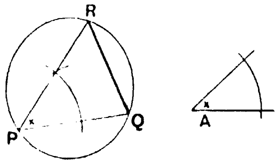


FIG. 341.

(ii) It may be necessary to produce RP backwards to meet the circle in S.

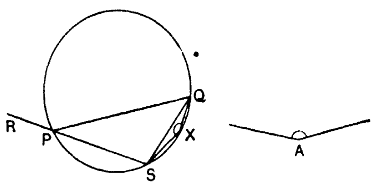


FIG. 342.

Join SQ and take any point X in the minor arc SQ.

$$\therefore \hat{SXQ} = 180^\circ - \hat{SPQ} \qquad \text{(SPQX cyclic quad.)}$$
$$= \hat{RPQ}$$
$$= \hat{A}.$$

$\therefore$ SQ cuts off the required segment.

(iii) If the line PR happened to be a tangent at P, then by Theorem 39, PQ cuts off the required segment.

The perpendiculars from the angular points of a triangle to the opposite sides are concurrent.

Let ABC be a triangle and BE, CF perp. to AC, AB respectively.

Join AO and produce to D.

We have to prove that AOD is perp. to BC.

Join FE.

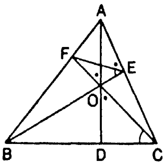


FIG. 343.

$\because$ the angles AEO and AFO are right angles;

$\therefore$ AEOF is a cyclic quadrilateral.

Also, since BFC and BEO are right angles,

$\therefore$ BFEC is a cyclic quadrilateral.

$\therefore\ A\hat{E}F = A\hat{O}F$ (AEOF cyclic)

$= C\hat{O}D$; (vertically opposite)

but $O\hat{C}D$ (*i.e.* $F\hat{C}B$) $= F\hat{E}B$ (BFEC cyclic)

$= 90° - A\hat{E}F$

$= 90° - C\hat{O}D$;

$\therefore\ O\hat{C}D + C\hat{O}D = 90°$,

$\therefore\ O\hat{D}C = 90°$,

i.e. AOD is perp. to BC.

The point O, where the perps. meet, is called the **orthocentre** of the triangle.

EXERCISE 30.

1. Draw a triangle ABC, $a=2{\cdot}6$, $b=4{\cdot}2$, $c=3$. Make $B\hat{C}D=62°$, $B\hat{A}D=118°$. Bisect AB and BC perpendicularly by lines meeting at O. Measure OB and OD. Explain your results

2. Draw a quadrilateral ABCD ; AB=4, BC=5, $\hat{B}=120°$, $\hat{A}=130°$, $\hat{C}=50°$. Draw the circumcircle of ABC, and explain why it passes through D. Measure the radius.

3. Draw a quadrilateral ABCD with AB=6·2, BC=5·4, CA=6·7, AD=5·5, DC=5·2 cms
Measure the angles ABC, ADC and thus determine whether ABCD is cyclic.

4. In a cyclic quadrilateral ABCD, if O is the centre of the circle and angles AOB, BOC, COD are equal to 115°, 82°, 59° respectively, determine the magnitudes of the angles of the quadrilateral.

5. ABCD is a cyclic quadrilateral with AB=AD. If $B\hat{C}D=114°$, find the magnitude of the angle ADO, if O is the centre of the circle (Fig. 344)

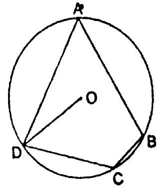


Fig. 344.

6. A is a fixed point on the circumference of a circle, AP any chord, M the mid-point of AP. Prove that the locus of M is a circle (Take S the centre of the ⊙, join SA and SM, prove that SMA is a right angle)

7. Two circles cut at A, B, and diameters AX, AY are drawn. Prove that X, B, Y are collinear

8. A ladder AB rests on the floor OX at B, and against the wall OY at A. If it slips, what is the locus of the centre P ?

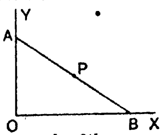


Fig. 345

9. If the two sides AB, DC of a cyclic quadrilateral ABCD, when produced, meet at P, and if PA=PD, prove that BC is parallel to AD.

10. If a regular hexagon is inscribed in a circle, what is the magnitude of one of the angles (such as ABC) in a segment external to the hexagon (Fig. 346).

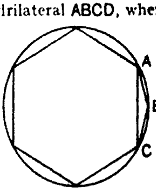


Fig. 346.

11. ABCD is a cyclic quadrilateral, and AC, BD meet in O, prove that the triangles ABO, DOC are equiangular and also the triangles AOD, BOC.

12. AOB is a diameter of a circle, centre O, and another circle is described on AO as diameter. AXY is a straight line meeting the circles in X and Y. Prove that AX = XY.

13. Two circles intersect in P and Q, and through P and Q are drawn lines APB, CQD, meeting the circles in A, B and C, D. Prove that AC is parallel to BD.

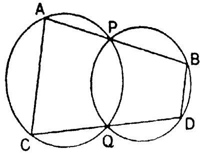


Fig. 347.

14. Prove that if a parallelogram can be inscribed in a circle, then it must be a rectangle.

15. If a trapezium can be inscribed in a circle, then the two non parallel sides must be equal.

16. Prove that the sum of the angles in the three segments of a circle external to an inscribed triangle is 4 right angles (Fig 348).

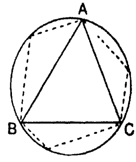


Fig. 348.

17. ABCD is a cyclic quadrilateral, AB and DC when produced meet at H, BC and AD at K. The circles round HBC and KCD meet again at P; prove that H, P, K are collinear.

18. ABC is a triangle in a circle and AD is perp. to BC. If AX is a diameter, prove that $B\hat{A}X = C\hat{A}D$.

19. Prove that the angle in a segment of a circle is greater than or less than a right angle, according as the segment is less than or greater than a semicircle (Fig. 349).

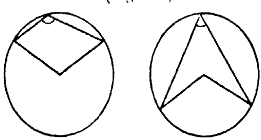


Fig. 349.

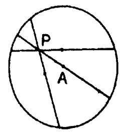


Fig. 350.

20. P is a fixed point within a circle whose centre is A. Prove that the middle points of all chords drawn through P lie on the circle whose diameter is AP (Fig. 350).

21. ABC is a triangle, D, E, F the feet of the altitudes. Prove that B, F, E, C lie on a semi-circle, that $A\hat{D}E = A\hat{C}F = 90° - A$; and dealing with $A\hat{D}F$ in a similar way, prove that AD bisects $F\hat{D}E$, that $A\hat{E}F = \hat{B}$ and the $\triangle^s$ AEF, BFD and CDE are equiangular to each other.

22. If XOY is a quadrant of a circle, and semicircles described on OX, OY as diameters, meet in A, prove that X, A, Y are collinear.

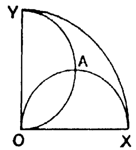


Fig. 151.

23. ABCD is any quadrilateral inscribed in a circle, the chord DE bisects the angle ADC, prove that the line BE bisects the angle between CB produced and BA

24. AB is a diameter and CD a chord of a circle AX and BY are drawn perp to CD produced. XA and BY (produced if necessary), cut the circle in E and F Prove that EX = BY, AX = FY.

(Prove that BX and FX are rectangles)

25. The bisectors of the exterior angles at Q and R of the triangle PQR meet at S Show that the centre of the circle through Q, R, S lies on the circumcircle of the triangle PQR

26. P is a point on the diagonal AC of a parallelogram ABCD. Circles are described about DPA and BPC Show that the other point of intersection of the circles lies on BD

27. Describe a $\triangle$ABC with $a = 5$, $b = 5{\cdot}5$, $c = 4{\cdot}2$ Find the orthocentre O. If the feet of the perps are D, E, F, measure $F\hat{D}E$ and verify that the angles FDO, EDO are equal

28. In the last question prove that the angles FDO and EDO are equal.

29. If O is the orthocentre of a triangle LMN, show that N is the orthocentre of OLM.

30. Construct a triangle ABC, given the altitude through A, the orthocentre, and the magnitude of angle B.

31. Given the base and vertical angle of a triangle, find the locus of its orthocentre

(If O is the orthocentre and BC the base, find the magnitude of the angle BOC)

32. If from any point P on the circumcircle of a triangle ABC perpendiculars PH, PK, PL are drawn to the sides BC, CA, AB respectively, prove that H, K, L are collinear. (HKL is called *Simson's Line.*)

(PKAL and PKHC are cyclic. Prove that PKH is the supplement of PKL.)

ANGLES, ARCS AND CHORDS.

THEOREM 34.

In equal circles (or in the same circle), equal angles at the centre stand on equal arcs, and conversely equal arcs subtend equal angles at the centre.

If the two circles ASB and CRD are equal in all respects and also the angles AXB, CYD, it follows that the one figure is a repetition of the other, and consequently

arc AB = arc CD.

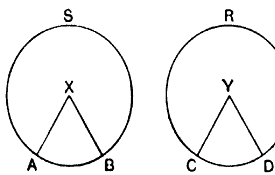


FIG 352

Similarly, if the arcs are equal it follows that the angles at the centre will be equal.

COR. *In equal circles (or in the same circle) equal angles at the circumference stand on equal arcs, and conversely, equal arcs subtend equal angles at the circumference.*

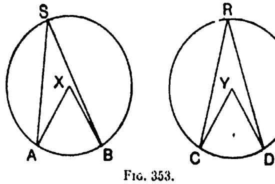


FIG. 353.

Since the angles at X and Y are each double of the angles at S and R, it follows that the Theorem is true for any angles, such as S and R, at the circumferences.

THEOREM 35.

In equal circles (or in the same circle), equal chords cut off equal arcs, and conversely the chords of equal arcs are equal.

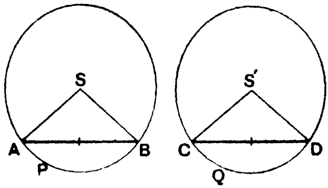


FIG. 354.

Let APB and CQD be two equal circles, S and S′ their centres.

(i) Given chord AB = chord CD, we have to prove

arc APB = arc CQD.

Construction. Join SA, SB, S′C, S′D.

Proof. In the △ˢASB, CS′D,

(i) AS = CS′,	(radii of equal ⊙ˢ)
(ii) SB = S′D,	,, ,,
(iii) AB = CD,	(hypoth.)
∴ △ASB ≡ △CS′D,	(3 sides)

and in particular, $\hat{ASB} = \hat{CS'D}$;

∴ arc APB = arc CQD. (Th. 34)

(ii) Similarly, if the arcs are equal, it follows that the angles ASB, CS′D will be equal, and consequently the triangles ASB, CS′D will again be congruent,

and chord AB = chord CD.

EXERCISE 31.

1. Prove that equal chords of a circle subtend equal angles at the centre.

2. AB and CD are two equal arcs of a circle, the sense of describing the circle from A to B being the same as that from C to D. Show that the chords BC and AD are parallel.

3. If two chords AB, CD of a circle are parallel, prove that the arcs AC, BD are equal (Join BC, Fig. 355.)

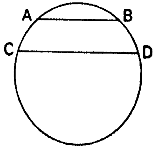


Fig. 355

4. ABC is a triangle inscribed in a circle and the bisector of the angle A meets the circumcircle in X. Prove that arc BX = arc XC

Hence show that if any number of angles be drawn in the same segment of a circle, the bisectors of these angles all pass through the same point.

5. If ABC is an arc of a circle and AC its chord, prove that it may be bisected by dividing AC equally at D, and erecting a perpendicular to AC at D.

6. If a triangle ABC is inscribed in a circle and a point X be taken on the arc BC remote from A, then if $B\hat{X}A = C\hat{X}A$, prove that AB = AC.

7. Two circles intersect at A and B and a chord XAY, variable in direction, passes through A. Prove that $X\hat{B}Y$ is constant in magnitude. (Join AB and note that it is constant in length, Fig. 356.)

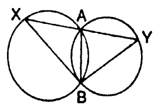


Fig. 356.

8. AKB and CKD are two perpendicular chords of a circle Q is a point on the arc AD such that arc DQ = arc AC Prove that BQ is a diameter. (Join AQ.)

9. AB, AC are equal chords of a circle Parallel straight lines AP, BQ are drawn cutting the circle again at P and Q. Prove that AQ is parallel to CP.

10. Three points A, B, C are taken on a circle: M, N are the mid-points of the arcs AB, AC respectively, show that the straight line MN forms an isosceles triangle with the chords AB, AC. (Join AM, AN.)

11. AB, AC are equal chords of a circle. P, Q are any two points in BC, and AP, AQ meet the circle again at R and S. Prove that a circle can be described about PQSR (Join SC.)

12. From a point A, on the circumference of a circle whose centre is O, two equal chords AB and AC are drawn CO and BO are joined, meeting the circumference in P and Q respectively. Prove that the chord AP is equal to the chord AQ.

13. Draw a triangle ABC in which AB=5·4 cms, BC=8·7 cms., CA=8·1 cms. Construct the circle that circumscribes the triangle.

From the mid-point of the arc BC draw the three perpendiculars to the sides of the triangle.

Prove that the feet of these perpendiculars lie on a straight line.

The Circumference of a Circle.

By using coins of various sizes it may easily be shown that the value of $\frac{\text{circumference}}{\text{diameter}}$ is constant, and that within the limits of experimental error

$$\frac{\text{circumference}}{\text{diameter}} = 3{\cdot}14159\ldots = 3\tfrac{1}{7} \text{ (approx.)}.$$

This value is called π, so that

$$\textbf{circum.} \text{ of } \odot = \pi \times \text{diameter}$$
$$= 2\pi r, \text{ where } r \text{ is the radius of } \odot.$$

Since equal arcs subtend equal angles at the centre of a circle, it follows that if two arcs AB, CD be taken on a circle, centre O,

$$\frac{\text{arc AB}}{\text{arc CD}} = \frac{A\hat{O}B}{C\hat{O}D},$$

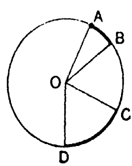


FIG. 357

and in particular,

$$\frac{\text{arc AB}}{\text{total circumference}} = \frac{A\hat{O}B}{360°}.$$

The Area of a Circle.

Draw a circle of radius r and cut it up into sixteen equal sectors, and arrange so that the sectors are alternately up and down, as shown in Fig. 358.

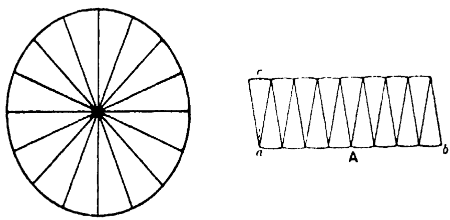


FIG. 358.

Area of $\odot$ = area of figure A.

Now A is approx. a parallelogram, and will be more nearly so, the greater the number of sectors taken ;

$\therefore$ area of A $= ab \times ac$, where ac is perp. to ab.

Also, the series of arcs contained in ab makes up half the circum. of the $\odot$;

$$\therefore\ ab = \pi r\ ;$$

$$\text{thus } \textbf{area of } \odot = ab \times ac = \pi r \times r$$

$$= \pi r^2.$$

A proof of this result is given on page 187.

EXERCISE 32.

(Assume $\pi = 3\frac{1}{7}$.)

1. Find the circumferences of circles with radii 7, 3·5 and 4·7 cms. respectively. (Give the answers to the nearest mm.)

2. Find the areas of circles with radii 7 ft., 8·5 metres (correct to the nearest sq. dm.), 9·1 cms respectively.

3. Find approximately the radii of circles, the circumferences of which are: (i) 250 cms., (ii) 200 inches, (iii) 5·2 yards.

4. Find approximately the radii of circles, the areas of which are: (i) 100 sq. yds; (ii) 58·56 sq metres, (iii) 48 sq. inches.

5. If the 28-inch wheels of a bicycle make 2500 revolutions each, how far does the machine travel? Answer to the nearest yard.

6. An arc of a circle of radius 5 inches subtends an angle of 50° at the centre. What is the length of the arc?

7. In sailing along a meridian from latitude 20° N to latitude 27° S., through what distance does a ship travel, if the radius of the earth is 3960 miles?

8. A sector of a circle of radius 5 inches contains an angle of (i) 45°, (ii) 65°; in each case what is its area?

9. A circular pond of water has a diameter of 170 feet. What is the area (in sq. ft.) of a path, 4 feet broad, which surrounds it?

10. What is the area of a circular ring whose inner and outer radii are 6 and 8·5 feet respectively? (Answer to the nearest sq. ft.)

11. A rectangle, 15 cms long and 8 cms broad, is inscribed in a circle; calculate the area of the circle, and thence deduce the sum of the areas of the four segments external to the rectangle.

12. What is the area of a square inscribed in a circle of diameter 6 inches?

13. The area of a circle of diameter d being taken to be $0{\cdot}785d^2$, find, correct to the nearest sq. yd., the area of the largest square which could be cut out of a circular field 10 acres in area.

14. The area of a circle of diameter d being taken to be $0{\cdot}785d^2$, find, correct to the nearest foot, the diameter of a circular plot, the area of which is $346\frac{1}{2}$ sq. yds.

15. The diameter of a circular brass disc, cut out of a sheet 5 mms. thick, is known to lie between 3·48 and 3·52 cms. If its volume is calculated on the assumption that the diameter is 3·5 cms., what is the greatest possible error in the result?

16. A semicircle of paper, two inches in diameter, is bent into the shape of a right circular cone by bringing the two bounding radii together. Show that the diameter of the base of the cone is one inch.

17. Find, as a decimal of a square foot, the area of the shaded quadrant ABD, and also the area of the unshaded part K, which together make up the square ABCD, whose side AB is 1 ft.

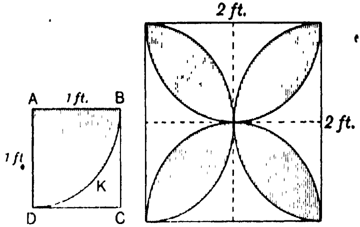


FIG. 359.

The second Fig. represents a window, each side of which is 2 ft., the shaded parts representing coloured glass, and the unshaded parts clear glass.

Find, by calculation, the area of the coloured glass and also the area of the clear glass.

If the amount of light passing through the coloured glass is half that passing through an equal area of clear glass, what fraction of the light that would pass through the window if it were all of clear glass is intercepted by using the coloured glass as there shown? Give your answer as a decimal fraction. (*Army.*)

TANGENCY.

Two simple curves usually intersect in two points, as in Fig. 360. It is noticeable that the points of intersection P, Q are not always the same distance apart; it is possible to draw the curves so that P, Q are indefinitely close to each other. When this is the case (Fig. 361), the curves are said to **touch**, and either curve is said to be a **tangent curve** to the other; the point P or Q is said to be the **point of contact.**

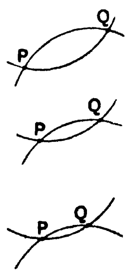


FIG. 360.

The word **Tangent** generally implies that one of the curves is a straight line, and we use the word as an abbreviation of *tangent line.*

Since a str. line can always be drawn through P, Q, it follows that **two curves which touch have a common tangent at the point of contact.**

The case which now concerns us is that of a line touching a circle, *i.e.* a line which meets the circle, but on being produced does not cut it.

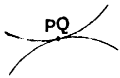


Fig. 361.

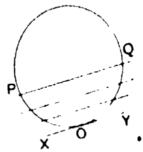


Fig. 362.

If the points P, Q approach one another so that they eventually coincide at O, then XOY (Fig 362), the limiting position of PQ, is the tangent to the circle at O.

DEF. A **tangent** to a circle is a straight line which cuts the circle at *two points indefinitely close together*, or in other words, the tangent meets the circle at one point, called the *point of contact*, and when produced either way does not cut it.

A **secant** is a straight line which cuts a circle; *e.g.* the line PQ in Fig. 363.

Instead of the points P and Q approaching one another, we may consider one of the points P to be fixed and imagine the line PQ to rotate round this point, while the other point Q approaches it and finally coincides with it, then PX, the limiting position of PQ, is the tangent at P

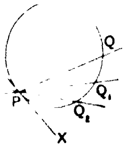


Fig. 363.

DEF. The **length of a tangent** from a point outside a circle is the distance between that point and the point of contact of the tangent with the circle.

THEOREM 36

The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.

Let O be the centre of the circle, OA a radius, and XAY a line perp. to OA. We have to prove that XAY is a tangent.

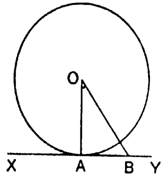


FIG. 361.

Construction. Take any point B on XY and join OB.

Proof. In the $\triangle OAB$, since $O\hat{A}B =$ rt. angle, and the sum of the three angles of a triangle is 2 rt. angles,

$$\therefore\ O\hat{B}A < \text{rt. angle};$$
$$\therefore\ O\hat{A}B > O\hat{B}A;$$
$$\therefore\ OB > OA$$
$$> \text{radius of the circle};$$

$\therefore$ the point B is outside the circle.

Similarly, any other point on XY, (except A), is outside the circle,

i.e. the line XY meets the circle at only one point, A,

i.e. XY is a tangent.

COR. At every point on a circle, one and only one tangent to the circle can be drawn.

(For only one perp. can be drawn to OA through the point A.)

NOTE.—The perp. to a tangent at its point of contact must pass through the centre of the circle. For, if not, there could be drawn at the point of contact a line perpendicular to the radius, and this would have to be a second tangent.

THEOREM 37.

From an external point two tangents can be drawn to a circle, and they must be equal in length.

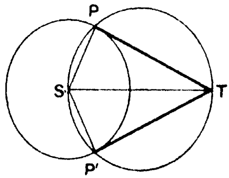


FIG. 365.

Let T be any point outside a circle, centre S.

We have to prove that two tangents can be drawn from T, and that they must be equal.

Construction. Describe a circle on ST as diameter, cutting the original circle at P, P′.

Join TP, TP′, SP, SP′.

Proof. SPT, SP′T are angles in a semi-⊙ ;

$$\therefore\ S\hat{P}T = 90° = S\hat{P}'T\ ;$$

thus the radii SP, SP′ are $\perp^{ar}$ to PT, P′T ;

$\therefore$ PT and P′T are tangents.

Again, in the right-angled △^{s}SPT, SP′T,

(i) SP = SP′, (radii of same ⊙)

(ii) hypot. ST is common ;

$\therefore$ △SPT ≡ △SP′T,

and in particular TP = TP′.

CONSTRUCTION 9.

To draw a circle touching the sides of a triangle, i.e. to draw the inscribed circle of a triangle.

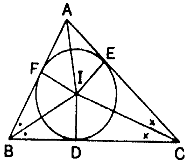


FIG. 386.

Let ABC be the triangle.

Bisect the angles B and C by lines meeting at I, and draw ID, IE, IF $\perp^{ars}$ to the sides.

In the right-angled $\triangle^s$ IDB, IFB,

$\widehat{IBD} = \widehat{IBF}$, (Constr.)

hypot. IB is common ;

$\therefore$ $\triangle IDB \equiv \triangle IFB$, (2 angles, 1 side)

and in particular, $ID = IF$.

Similarly, $\triangle IDC \equiv \triangle IEC$,

and $ID = IE$;

$\therefore$ $ID = IF = IE$.

Also the angles at D, F, E are rt. angles, $\therefore$ the circle, centre I and radius ID will touch the sides at D, F, E.

COR. Join IA, then in the right-angled $\triangle^s$ IAF, IAE,

$IF = IE$,

hypot. IA is common ;

$\therefore$ $\triangle IAF \equiv \triangle IAE$;

$\therefore$ $\widehat{IAF} = \widehat{IAF}$,

i.e. the three bisectors of the angles of a triangle are concurrent.

The circle is called the **Inscribed Circle** of the triangle, or sometimes the In-circle.

I is the centre of the inscribed circle, and is called the in-centre.

The **Escribed Circle** of a triangle is one which touches one of the sides of the triangle and the other two sides produced.

CONSTRUCTION 10.

To draw the escribed circle of a triangle.

Let ABC be a triangle with the two sides AB, AC produced.

Bisect the external angles at B and C by BI_1, CI_1.

Draw I_1F, I_1D, I_1E perp. to AB produced, BC, and AC produced, respectively.

As in the last construction, it can be proved that

$$I_1F = I_1D = I_1E,$$

and the angles at F, D, E are right angles.

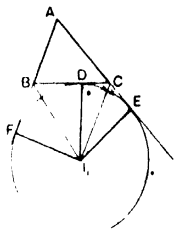


FIG. 367.

$\therefore$ a circle can be drawn with I_1 for centre to touch BC, and AB, AC produced, at D, F, E.

COR. 1. If I_1A is joined, it can be proved that I_1A bisects angle A.

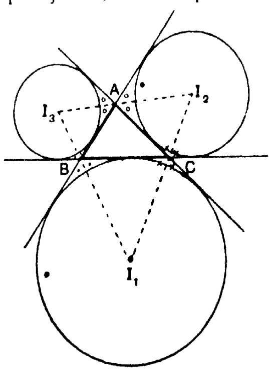


FIG. 368.

COR. 2. If I_1, I_2, I_3 are the three e-centres, then the sides of the $\Delta I_1I_2I_3$ pass through A, B, C.

CONSTRUCTIÓN 11.

To draw a triangle equiangular with a given triangle and with its sides touching a given circle.

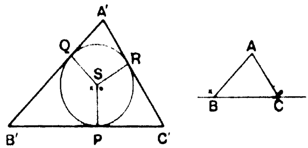


FIG. 360.

Let ABC be the given triangle.

Draw SP any radius of the given circle.

Make $\quad \hat{PSQ}$ = the supplement of $\hat{B}$,

$\hat{PSR}$ = the supplement of $\hat{C}$.

Draw tangents at P, Q, R, meeting at A′, B′, C′.

$\because \hat{P} = 90°$ and $\hat{Q} = 90°$, $\therefore$ SPB′Q is cyclic;

$\therefore \hat{PSQ}$ = the supplement of $\hat{B'}$;

but $\hat{PSQ}$ = the supplement of $\hat{B}$;

$\therefore \hat{B'} = \hat{B}$.

Similarly $\hat{C'} = \hat{C}$,

and $\therefore \hat{A'} = \hat{A}$.

EXERCISE 33.

1. Draw two tangents to a circle, radius 2·4 cms., from a point distant 4 cms. from the centre, and measure their lengths.

2. Draw $A\hat{B}C = 40°$; in BA take S, S′ such that BS = SS′ = 2·8 cms. With centres S, S′ draw circles touching BC, measure the distance between the points of contact.

3. Draw a circle, radius 2·6 cms, draw AT a tangent at A and AB a chord such that $T\hat{A}B = 40°$, draw BC a diameter; measure BA and $A\hat{C}B$.

4. If AB, AC are two tangents to a circle from an external point A, and $B\hat{A}C = 80°$, what is the angle between the two radii OB, OC? If the angle between the radii is 72°, what is the angle between the tangents?

5. Draw a circle having a radius of 3·6 cms From a point P, distant 6 cms. from the centre draw a line touching the circle at L. Calculate the length of PL.

6. Calculate the length of the chord joining the points of contact of the tangents drawn to a circle of radius 4 cms from a point distant 7 cms. from its centre

7. A quadrilateral PQRS (Fig 370) touches a , centre O, at A, B, C, D. If $A\hat{O}B = 62°$, $B\hat{O}C = 84°$, $C\hat{O}D = 110°$, calculate the angles P, Q, R, S.

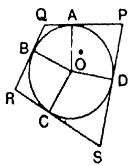


FIG 370

8. Two concentric circles are drawn whose radii are respectively 4·8 cms. and 6 cms. A straight line HK touches the inner circle at H and cuts the outer one at K. Calculate the length of HK.

9. Draw an isosceles triangle whose base, BC, is 3 inches, and vertical angle A is 36°. Construct a circle to pass through A and touch BC at C. Let the circle cut AB at P. Measure AP.

(If O is the centre, OA = OC and $O\hat{C}B = 90°$.)

10. Draw two perpendicular lines AB, BC. Make AB = 1½ inches, BC = 2½ inches. Construct a circle to pass through C and to touch AB at A. Measure the radius of this circle.

11. Draw $A\hat{O}B=50°$ and construct a circle of radius 2·8 cms. to touch OA, OB at X and Y. Measure OX.

(The centre must be on the bisector of $A\hat{O}B$ and 2·8 cms. from OX.)

12. Draw a triangle ABC so that $a=6·7$ cms., $b=7·6$ cms., $c=5·8$ cms. Construct a circle to touch AB, AC, and having its centre on BC. Measure the radius.

13. P is a point 6·5 cms. from a line XY. Draw a circle of radius 4 cms. to pass through P and touch XY. If the circle touches XY at A, measure PA.

14. Two lines AX, AY, meet at an angle of 40°; along AX two points B and C are taken so that AB=1·5 ins., and AC=2·7 ins. Construct a circle that shall pass through B and C, and have its centre 1 inch from AY. How many such circles are there? Measure their radii.

15. Construct a triangle ABC such that BC=3·5 cms., CA=7·5 cms., and AB=8·5 cms., construct the escribed circle of the triangle opposite the angle A, and measure its radius.

16. A thin hollow cone, of height 4 inches, stands on a circular base, of radius 3 inches. A spherical ball, of the largest size possible, is placed within the cone. Draw a section of the cone and the ball by a plane through the axis of the cone, and measure the radius of the ball.

17. Two circles of radii 4·4 cms. and 3·2 cms., touch a straight line (on the same side of it) at X and Y. If XY=5·7 cms., calculate the distance between the centres of the circles.

18. If AB and AC are the two tangents to a circle from an external point A, and O the centre, prove that

(i) $A\hat{B}C=A\hat{C}B$,

(ii) OA bisects BC at right angles.

19. At the points P, Q, R, S of a circle, tangents are drawn forming a quadrilateral ABCD. Prove that AB+CD=BC+AD (Fig. 371).

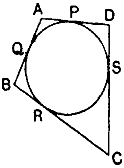


FIG. 371.

20. Show how to draw a tangent to a given circle, parallel to a given straight line.

21. A series of tangents are drawn to a circle, and points are taken along the tangents at a given distance from the points of contact; prove that these points lie on a circle.

22. Show how to draw two circles touching three straight lines, two of which are parallel.

23. If two circles are concentric, prove that the length of the tangent (PA) drawn from any point on the outer circle to the inner circle is constant in length (Fig. 372)

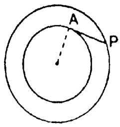


FIG. 372

24. If any two circles cut at A and B, prove that the angle between the two tangents at A equals the angle between the two radii through A.

25. AP and BQ are two fixed parallel tangents to the circle ARB, a third tangent PRQ, at any point R, cuts them in P and Q, prove that PQ subtends a right angle at the centre of the circle

26. PA is a tangent to a circle and PBC a secant If O is the centre of the circle and OD is drawn perp to BC, prove that

$$A\hat{D}P = A\hat{O}P$$

27. Any two circles, centres S, S′, are drawn on the same side of a straight line, touching it at A, B; SS′ produced meets AB produced at T, and $S\hat{T}P$ is made equal to $S\hat{T}A$; prove that TP will touch both the circles

28. Parallel chords AC, BD are drawn through two fixed points A, B on a fixed circle Prove that CD always touches a fixed concentric circle.

29. If two circles cut *orthogonally* (so that the tangents at a point of intersection are at right angles), prove that the sum of the squares of the radii is equal to the square on the distance between the centres.

INSCRIBED AND CIRCUMSCRIBED REGULAR POLYGONS.

To inscribe a regular polygon in a circle.

If the figure has n sides the angle subtended at the centre by any side will be $\frac{360^\circ}{n}$.

Draw a set of radii containing this angle between every adjacent pair, then the chords joining the extremities will give the required figure. Explain why the figure is both *equilateral* and *equiangular*.

An Equilateral Triangle should be inscribed by drawing three radii at angles of 120°, etc.

A **Square** is inscribed by drawing two diameters at right angles.

A **Hexagon** is obtained by drawing three diameters at angles of 60°.

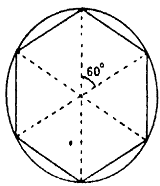


FIG. 373

In this case, since each of the $\triangle^s$ is equilateral, the figure can be obtained by marking off six chords equal to the radius of the circle.

An **Octagon** (Fig. 374) is described by drawing four diameters at angles of 45°.

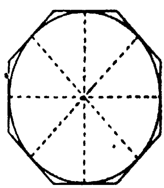


FIG. 374.

To circumscribe a regular polygon about a circle.

Proceed as above and draw tangents at the extremities of the radii.

Explain why the figure so obtained is both equilateral and equiangular.

To find the area of a circle.

Let AB, BC, ... be the sides of a regular polygon circumscribed about a circle, O the centre of the ⊙, and OX, OY, ... the radii to the pts. of contact of the sides.

Area of polygon $= \triangle OAB + \triangle OBC + \dots$

$$= \frac{1}{2}AB \cdot OX + \frac{1}{2}BC \cdot OY + \dots$$

$$= \frac{r}{2}(AB + BC + \dots)$$

$$= \frac{r}{2} \times \text{perim. of polygon.}$$

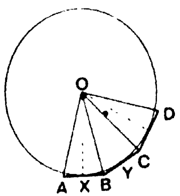


FIG. 375.

Now suppose the number of sides of the polygon increases indefinitely, then the perimeter of the polygon differs from the circumference of the circle by a quantity which is indefinitely small, and in the limit, the perimeter of the polygon equals the circum. of the circle;

$$\therefore \text{ area of circle} = \frac{r}{2} \times 2\pi r$$

$$= \pi r^2.$$

EXERCISE 34.

1. Inscribe an equilateral triangle in a circle of 1 inch radius, and measure its sides.

2. Inscribe an equilateral triangle in a circle of 3 cms. radius, and measure its sides.

3. Circumscribe a circle of radius 2·5 cms. with a square. Measure its diagonals.

4. Inscribe a regular hexagon ABCDEF in a circle of radius 2·6 cms. Measure AC and find the area of the hexagon.

5. Circumscribe a circle of radius 1·2 inches with a regular hexagon, and measure the sides.

6. Circumscribe a circle of radius 1·4 inches with a regular octagon, and measure the diagonals.

7. Inscribe a regular octagon in a circle of 5·2 cms. diameter, and measure the sides. Also find the area of the octagon.

8. Inscribe 4 equal circles in a square of side 5 cms., and in each case measure AB, the distance between two centres (Figs. 376, 377).

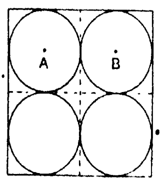


Fig. 376.

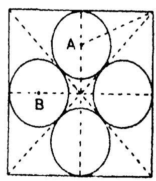


Fig. 377.

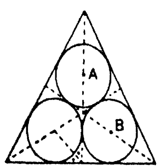


Fig. 378.

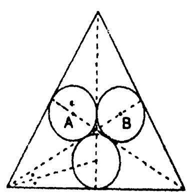


Fig. 379.

9. Inscribe 3 equal circles in an equilateral triangle of side 6 cms., and in each case measure AB (Figs. 378, 379).

10. Inscribe 6 equal circles in a regular hexagon of side 2·8 ins., and in each case measure AB (Figs. 380, 381).

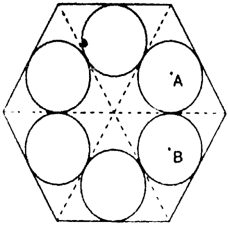


Fig. 380

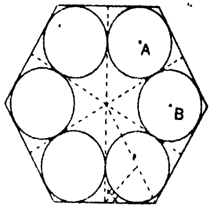


Fig. 381

11. Draw 3 equal semicircles in an equilateral △ of side 5 inches. Measure the distance between two centres AB (Fig. 382)

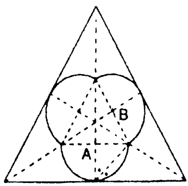


Fig. 382.

12. Draw 4 equal semicircles in a square of side 2·5 ins. Measure AB, the distance between two centres (Fig. 383)

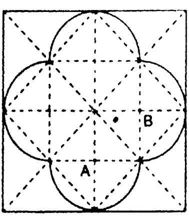


Fig. 383

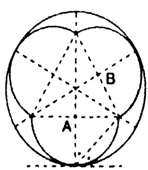


Fig. 384.

13. Draw 3 equal semicircles in a circle of radius 3·5 cms., and measure AB, the distance between two centres (Fig. 384).

THEOREM 38.

If two circles touch one another, their centres and the point of contact are collinear.

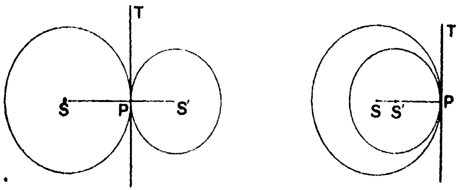


FIG. 385.

Let the two circles, centres S, S′, touch at P, having a common tangent PT.

We have to prove that S, S′, P are collinear.

Construction. Draw a straight line perp. to PT.

Proof. Since PT is a tangent to the circle centre S, it follows that the perp. to PT passes through S.

Similarly, since PT is a tangent to the circle centre S′, the perp. to PT passes through S′.

∴ S, S′, P are in the same straight line, namely, the perp. to PT at P.

COR. 1. AB is a diameter of a ⊙ and P any point on AB produced. A circle, centre P and radius PB, will touch the original circle at B

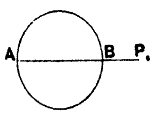


FIG. 386.

COR. 2. The distance between the centres of two circles which touch is either the sum of the radii or the difference of the radii.

By the Principle of Symmetry, since SS′ is the line of diameters, and the circles are symmetrical about this line, it follows that the common points A and B are symmetrical points with respect to the axis SS, ∴ AB is bisected at right angles at P by SS′.

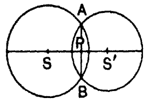


FIG. 387.

If now the circles move apart, A and B gradually approach and eventually coincide at P, since AP and BP are always equal; while APB becomes the tangent at P and remains perp. to SS′.

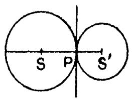


FIG. 388.

We have a similar result if the circles move in the opposite directions until they reach a position of internal contact.

EXERCISE 35.

1. Draw a circle of $1\frac{1}{2}$ inch radius, and describe a number of touching circles of radius $\frac{1}{4}$ inch, both inside and outside the original circle. What is the locus of A and of B?

2. Construct three circles of radii 2·7 cms., 3·4 cms., and 4·2 cms. respectively, so that each circle touches the other two externally, and draw the tangents to these circles at the three points of contact.

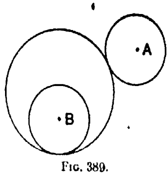


Fig. 389.

3. If three circles mutually touch each other externally, and the distances between their centres are 5, 6, 7 cms respectively, calculate the radii.

4. Draw a circle of radius 1·2 inches, and then another circle of radius 2·2 inches to touch and enclose it. Draw several of these larger circles and indicate the locus of their centres.

5. Draw a circle of radius 3·5 cms., and place in it another circle of radius 1 cm., so as to touch the first circle. Draw several such small circles and thus find the locus of their centres.

6. Draw a circle of radius 2·6 cms., and in it place several circles of radius 1·3 cms. to touch the large circle. What is the locus of the centres of the small circles?

7. In a semicircle (Fig. 390) of radius 8 cms. inscribe a circle of radius 2 cms. to touch the arc and diameter of the semicircle. Measure OA.

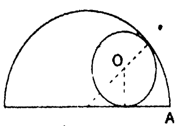


Fig. 390.

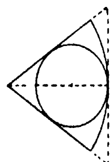


Fig. 391.

8. In a sector of a circle of radius 4·5 cms. and angle 72° inscribe a circle. Measure the radius (Fig. 391).

9. If two circles touch at A, and a straight line through A meets the circles in B and C, prove that the diameters through B and C are parallel (Fig. 392).

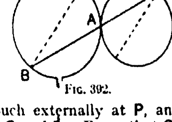


Fig. 392.

10. Two circles touch externally at P, and a common tangent touches the circles at Q and R. Prove that QPR is a right angle.

11. Two unequal circles touch externally at P. The arc AP is one-fifth of the circumference of one circle. Prove that the chord AP, when produced, cuts off one-fifth of the circumference of the other circle.

12. Two circles with centres H and K touch one another externally. A straight line touches the circle with centre H at B, and the circle with centre K at C. Prove that HK is a tangent to the circle on BC as diameter

13. Draw a circle of radius 2 inches and another circle of radius 1 inch to touch the former internally at a given point O Prove that, if any chord OAB is drawn through O to meet the smaller circle in A and the larger in B, then A is the middle point of OB

14. Two circles, on the same side of the line PX, touch it at P, and a straight line PAB is drawn cutting the circles at A and B. Prove that the tangents at A and B are parallel.

15. A variable circle, centre A, touches externally two fixed circles with centres P and Q Prove that AP ~ AQ is constant

16. Two circles ABC, ADE, of which the centres are F and G, touch at the point A, and a straight line BAD is drawn through A cutting the circles in B and D Prove that FB is parallel to GD.

It is sometimes necessary to draw arcs of different circles so that they form a continuous curve.

In Fig. 393, where two circles, centres A and B, touch at D,

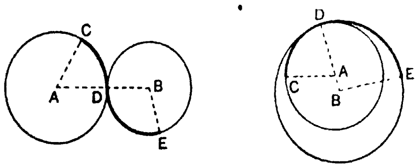


Fig. 393.

there is no break or jerk in passing from the arc CD on one circle to the arc DE on the other. Examples of this will be found in the curves on railroads and in the arches and the tracery of windows in architecture.

EXERCISE 36.

1. Draw a line AB 3 cms. long and take AC=CD=1 cm. On AB as diameter describe a circle, and then draw continuous semicircles (Fig. 394).

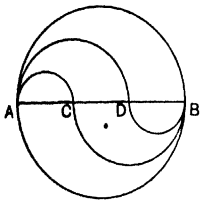


Fig. 394.

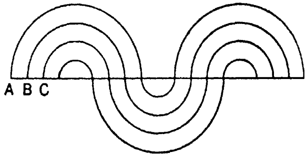


Fig. 395.

2. Take a number of equidistant points A, B, C, ... and draw continuous semicircles (Fig. 395).

3. Draw a figure of 8 (Fig. 396).

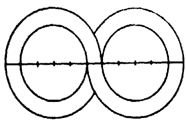


Fig. 396.

4. A railway goes from X to Y; XA and CY are straight lines, AB, BC continuous arcs of circles with centres at P and Q.

(i) AP=1000 yds., QB=1500 yds., $\hat{APB}=40°$, $\hat{BQC}=50°$.
Draw the diagram (Fig. 397) to scale and measure the chords AB, BC.

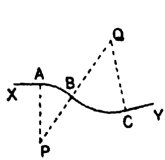


Fig. 397.

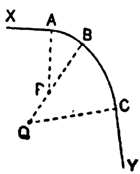


Fig. 398.

(ii) AP=1200 yds., QB=1450 yds., $\hat{APB}=35°$, $\hat{BQC}=42°$.
Draw the diagram (Fig. 398) to scale and measure AC.

5. The arcs AB, BC, CD, DE (Fig. 399) form the outline of an "ogee"-shaped arch, the upper arcs touching at C. AF=GE=3 cms., FG=2 cms., $A\hat{G}B=55°$.

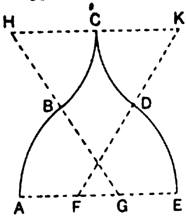


FIG. 399.

Draw the diagram to scale and measure the chords AB, BC.

6. An oval is formed of four circular arcs, which have common tangents at their points of contact (*see* rough sketch, Fig 400). The

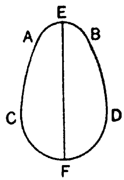


FIG. 400.

length EF is 4·6 inches. The radius of the arc AB is 1·1 ins., that of CD 1·6 ins., that of BD and of AC 4·2 ins.

Construct the oval full size. (*Army*)

7. The large circle (Fig. 401) has a radius of 4 cms.; the circles with centres at A, B, C have a radius of 1·6 cms.; the circles with centres at X, Y, Z have a radius of 1·3 cms. Draw the figure.

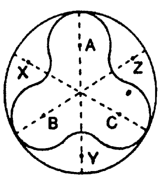


FIG. 401.

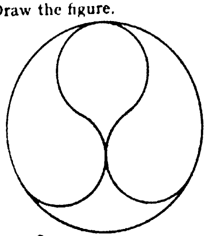


FIG. 402.

8. Draw three arcs (Fig. 402) of equal radii 1·8 cms. touching a larger circle internally.

9. Given that ABC is an equilateral triangle of side 3 cms., draw a diagram similar to Fig. 403, which contains 12 arcs of equal radius.

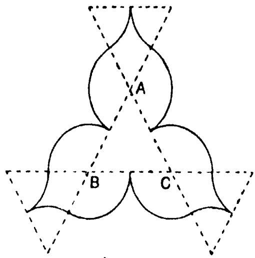


FIG. 403

10. (i) Two straight lines meet and terminate at O, forming an angle of 40° (Fig. 404). A circle S touches the two lines as in the figure. Draw the corresponding figure when the radius of the circle S is $1\frac{1}{4}$ ins.

(ii) Draw two other circles, each of them to touch S and the two lines (not produced past O). Measure the distance between the centres of the two circles so drawn.

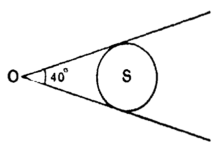


FIG. 404.

11. Draw two circles of radii 2 ins. and 3 inches with their centres 4 inches apart, and then draw a circle of radius 1 inch which touches each of them externally. Measure the distance between the pts. of contact.

(Let A and B be the centres of the given circles. With A and B as centres and radii 3 and 4 inches respectively, describe arcs cutting at O. Then O is the required centre.)

12. A is the centre of a circle of 1·5 inches radius; B is the centre of another circle of 0·8 inch radius; and AB = 4·0 inches. Construct a circle of 1·2 inches radius to touch the other two circles externally. Measure the distance between the points of contact.

13. P is a point on the circumference of a circle of centre O and radius $1\frac{1}{4}$ ins. Q is taken so that $P\hat{O}Q = 40°$ and OQ = 3 ins. Construct a circle to touch the given circle at P and to pass through Q. Measure the radius of this circle.

(Join PQ; produce OP to A; make $P\hat{Q}A = Q\hat{P}A$. Then A is required centre.)

14. Q is a point on the circumference of a circle, centre C and radius 2·4 cms. $Q\hat{C}P = 125°$ and CP = 4·3 cms. Draw a circle to pass through P, touch the given circle at Q and enclose it. Measure the radius.

15. A given circle with centre O has a radius 1·5 ins. P is a point so that OP=3·1 ins. Draw a circle of radius 1 inch to pass through P and touch the given circle externally at B. Measure BP.

(Draw a concentric circle with radius 2·5 ins.; with P for centre and radius 1 inch, draw an arc cutting this circle at A. Then A is the centre required.)

16. A circle with centre O has a radius of 2·8 cms. and P is a point inside it, such that OP=2 cms. Draw a circle of radius 1 cm to pass through P and touch the given circle internally at A. Measure AP.

17. O is the centre of a circle of radius 1 inch, and A is a point at a distance 2·5 inches from O. Construct a circle touching the given circle and also touching AO at A. Measure its radius.

18. Draw an equilateral triangle ABC having its sides 3 inches in length; with centre A draw a circle of radius 1 inch, and then construct a circle touching this circle externally, and also touching BC at B. Measure its diameter.

(If AD is a median of △ABC, produce DA to meet circle in E, join EB, cutting the circle at F; join AF, and produce to meet the perp. to BC at B in O. Then O is the required centre.)

19. Draw a circle, radius 2·4 cms., having its centre 4 cms. from a given straight line, and take a point on the line 7 cms from the centre of the circle. Draw another circle to touch the line at the given point and to touch and contain the circle. Measure its radius.

20. AB is a given straight line and C a point whose perp. distance from AB is 4·2 cms. C is the centre of a circle of radius 2 cms. Draw a circle of 1·8 cms to touch the line AB and the given circle externally, and measure the distance between the points of contact.

(On the same side of AB as C, draw XY ∥ AB and 1·8 cms. from it. With centre C and radius 3·8 cms. draw an arc cutting XY at O. Then O is the required centre.)

21. C is the centre of a circle of radius 1 cm. and AB a line whose distance from C is 5·4 cms. Describe a circle of radius 3·5 cms to touch the line AB on the same side as C, and to enclose the given circle. Measure the distance between the points of contact.

22. A, B, C are three points such that AB=3·5 cms., BC=4 cms., CA=4·3 cms., and three equal circles of radius 1·4 cms. are drawn with A, B, C for centres. Draw a circle to touch these three circles and enclose them. Measure its radius.

23. OPQ is a triangle with OP=5·2 cms., PQ=4·8 cms., QO=4·2 cms. Draw three circles with centres at O, P, Q, so that each touches the other two externally. Measure the radii.

(Bisect angles POQ, PQO by lines meeting at A. From A draw perps. AB, AC, AD to OP, OQ, PQ respectively. Then B, C, D are the points of contact.)

24. Draw two circles, centres A and B, with radii 1·8 cms. and 3·2 cms. respectively, so that AB=6 cms. Take a point P on circle B so that $A\hat{B}P=40°$. Draw another circle to touch both the circles externally, the larger circle at P. Measure the radius.

(Join BP and produce; draw AC ∥ BP to meet circle A at C. Join CP, meeting circle A at D. Join AD and produce to meet BP produced at O. Then O is the required centre.)

THEOREM 39.

The angles which a tangent to a circle makes with a chord drawn through the point of contact are equal to the angles in the alternate segments of the circle.

Let TPT′ be a tangent to the circle ABC and PA any chord through P, the point of contact.

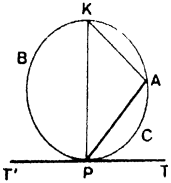


FIG. 405.

Draw any angles in the segments ABP and ACP, then we have to prove

$\widehat{APT}$ = the angle in the segment ABP,

$\widehat{APT'}$ = „ „ ACP.

(i) *Construction.* Through P draw a diameter PK and join AK.

Proof. $\widehat{KAP}$ = rt. angle, (angle in a semicircle)

∴ $\widehat{AKP} + \widehat{APK}$ = rt. angle,

i.e. $\widehat{AKP}$ = complement of $\widehat{APK}$;

but $\widehat{APT}$ = „ „ $\widehat{APK}$, ($\widehat{KPT}$ = rt. angle)

∴ $\widehat{APT}$ = angle in segment ABP.

(ii) *Construction.* Take any point C in the arc ACP and join CA, CP.

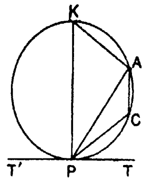


FIG. 406.

Proof. $\hat{APT'}$ = supplement of $\hat{APT}$,

and $\hat{ACP}$ = " " $\hat{AKP}$, (ACPK cyclic)

but $\hat{APT} = \hat{AKP}$, (just proved)

$\therefore$ $\hat{APT'}$ = angle in segment ACP.

This result may be proved from the limiting case of the cyclic quadrilateral in Theorem 32.

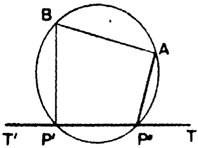


FIG. 407.

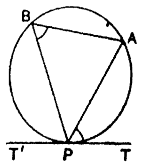


FIG. 408.

With a quad. as in Fig. 407, we have

$$\hat{AP''T} = \hat{B}.$$

In the special case where P′ and P″ coincide at P (Fig. 408), this becomes $\hat{APT} = \hat{B}$.

EXERCISE 37.

1. AT is a tangent to a circle at A (Fig. 409). If $C\hat{B}A = 42°$, $B\hat{A}T = 20°$, find the other angles of the figure.

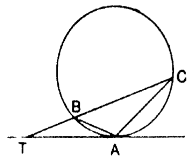


FIG. 409.

2. In Fig. 409, if AC = AT and $C\hat{A}T = 110°$, find the other angles of the figure.

3. In Fig. 410, if $E\hat{B}D = 30°$, $D\hat{B}C = 10°$, and ABC is a tangent and BF a diameter, find the other angles of the figure.

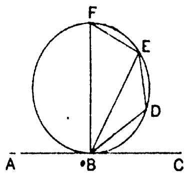


FIG. 410.

4. If ABC is a triangle in which $\hat{A} = 42°$, $\hat{B} = 58°$, and the inscribed circle touches the sides BC, CA, AB in D, E, F respectively, find the angles of the triangle DEF.

5. If an arc of a circle is given, how would you draw the tangents at its extremities without finding the centre of the circle?

6. Two circles touch internally at P and PQ is a common tangent. PAC, PBD are chords cutting one circle in A, B, and the other circle in C, D. Prove that AB is parallel to CD.

7. Two circles touch externally at P, and straight lines APD, BPC are drawn meeting one circle in A, B, and the other circle in C, D. Prove that AB is parallel to CD.

8. CT is a tangent to a circle meeting it at C, and AB is a chord parallel to CT. Prove that C is the middle point of the arc ACB.

9. AB is a diameter of a circle, T any point on the circumference and TN a perpendicular to AB. Prove that AT and BT bisect the angles between TN and the tangent at T.
(Note that angle ATB $=90°$.)

10. Two circles have a common chord AB; through A a tangent is drawn to one circle, meeting the other at P, and PB produced meets the first circle at Q. Prove that AQ is parallel to the tangent at P.

11. Two circles intersect at A and B. At A tangents to the circles are drawn, meeting the other circles at X and Y. Show that BA bisects the angle XBY.

12. Two circles touch internally at A, and a chord PQRS cuts them; prove $P\hat{A}Q = R\hat{A}S$.

13. ABCD is a cyclic quadrilateral; AD, BC meet at E; prove that the tangent at E to the circle CDE is parallel to AB.

14. H, K are two points in the sides AB, AC of a triangle such that HBCK is cyclic; prove that the tangent at A to the circumcircle of ABC must be parallel to HK.

15. H, K are two points in the sides AB, AC of a triangle such that HK is parallel to BC; prove that the circumcircles of ABC, AHK touch.

16. AC is a chord of a circle and is bisected at B. BD is drawn perp. to AC to meet the circle in D and DE is drawn perp. to the tangent at A. Prove that DE = DB.

17. Two circles touch internally at A and AT is the common tangent. A chord BC of the larger circle touches the smaller one at D. Prove that $B\hat{A}D = D\hat{A}C$.

18. Tangents to a circle at the extremities of two perpendicular chords form a cyclic quadrilateral.

19. From a point P a tangent PA is drawn to a circle, and a secant PBC cutting the circle in B and C. The bisector AQ of the angle BAC cuts BC at Q. Prove that PQ = PA.

20. QR is a chord of a circle, TR is the tangent at R; a straight line through Q perpendicular to this tangent meets it in T and the circumference of the circle again in P, PM is the perpendicular from P on QR. Prove that the angles QPM, TPR, TMR, TRM are all equal.

21. Draw a circle touching the side AB of a triangle ABC at A, and passing through C.
Draw another circle touching CA at C and passing through B, cutting the first circle at P. Prove that the angles PAB, PBC, PCA are all equal.

COMMON TANGENTS.

The number of lines which can be drawn to touch both of two given circles depends on the relative positions of those circles.

In the general case, four such tangents can be drawn, two being called *external* or *direct* common tangents, and the other two, which pass in between the circles, being *internal* or *transverse* common tangents.

As the circles alter their relative positions the number of common tangents may be 3, 2, or 1; and finally, if one circle is entirely inside the other, there is no common tangent.

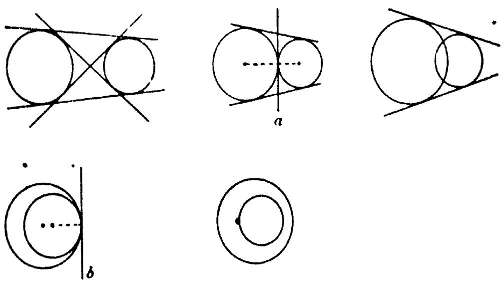


FIG. 411.

A common tangent such as *a* or *b* is obviously obtained by drawing a line perp. to the line joining the centres of the circles.

CONSTRUCTION 12.

To draw two external common tangents to two given circles.

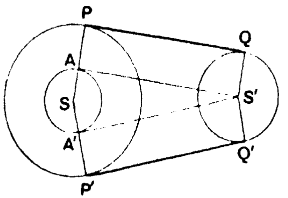


FIG 412

Let S and S′ be the centres of the two given circles, PP′, QQ′, and a and b their radii.

Construction. With centre S and radius $a-b$, draw a circle AA′.

By drawing a circle on SS′ as diameter, construct the two tangents S′A, S′A′, from S′ to this circle.

Join SA, SA′ and produce to meet the larger circle in P, P′.

Draw S′Q, S′Q′ ∥ SP, SP′ respectively.

Join PQ, P′Q′.

Then PQ, P′Q′ are the two external common tangents.

Proof. $SA = a - b, \quad SP = a;$

$\therefore \; AP = SP - SA = b = S'Q,$

and AP is ∥ to S′Q; (constr.)

∴ PAS′Q and similarly P′A′S′Q′ are ▱^ms^.

But $P\hat{A}S'$ is a right angle; (S′A $\perp^{ar}$ SA)

∴ PAS′Q is a rectangle,

i.e. $A\hat{P}Q$ and $P\hat{Q}S'$ are right angles;

∴ PQ is a tangent to both circles.

Similarly P′Q′ is a common tangent.

CONSTRUCTION 13.

To draw two internal common tangents to two given circles.

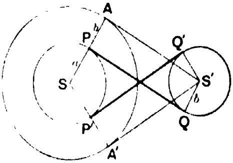


FIG 413.

Let S and S' be the centres of the two circles PP', QQ', and a and b their radii.

Construction. With centre S and radius $a+b$ describe a circle AA', and draw two tangents S'A, S'A' to it from S'.

Join SA, SA' meeting the circle PP' in P and P'.

Draw S'Q, S'Q' ∥ AS, A'S respectively.

Join PQ, P'Q', then PQ, P'Q' are the internal common tangents.

Proof. PA = SA − SP = $(a+b) - a = b$ = QS', and PA is ∥ to QS';

∴ PAS'Q and similarly P'A'S'Q' are ▭^{ms}.

But $\hat{\text{PAS'}}$ and $\hat{\text{P'A'S'}}$ are right angles;

∴ PAS'Q and P'A'S'Q' are rectangles,

i.e. SPQ (or APQ) and S'QP are right angles;

∴ PQ is a tangent to both circles.

Similarly P'Q' is a common tangent.

EXERCISE 38.

1. Draw two circles, radii 3·7 cms and 1·4 cms., with their centres 7 cms apart, and draw their two external common tangents. Measure the length of the tangent.

2. Draw and measure the internal common tangents in Qn. 1.

3. Repeat Qn. 1 with the numbers 4·2 cms, 2·2 cms., 8 cms.

4. Draw and measure the internal common tangents in Qn. 3.

5. Two circles, with centres 5 cms. apart, have radii of 3·7 and 2·3 cms., and cut at A and B.

Draw the external common tangents and let AB produced meet one of these tangents in P. Measure AP.

6. Two circles with radii 2 and 4 cms. have their centres 8 cms. apart. *Calculate* the lengths of their external and transverse common tangents.

7. Two circles, with radii a and b, have their centres a distance c apart. Prove that the difference of the squares of an external and internal common tangent equals $4ab$.

RECTANGLES.

We have already seen that the area of a rectangle ABCD, with length a cms. and breadth b cms., equals ab sq. cms.

Such a rectangle is said to be contained by two adjacent sides AB and AD, and

area of rect. ABCD = AB . AD ;

this latter expression is frequently used for the expression rect. ABCD.

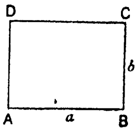


FIG. 414.

In the same way, the area of the square described on AB is represented by AB^2.

It is sometimes convenient to refer to the rectangle ABCD as AC or BD.

Illustration of Algebraical Identities.

FORMULA I.

$$k(a+b+c+d) = ka + kb + kc + kd.$$

Take a straight line AB, a ins. long; produce it to C so that BC $= b$ ins.; to D so that CD $= c$ ins.; to E so that DE $= d$ ins.; and so on.

Draw AP $\perp^{ar}$ to AE and k ins. long.

Draw through B, C, D, E lines $\parallel$ AP and through P a line $\parallel$ AE.

It is easily shown that all the figures formed are rectangles.

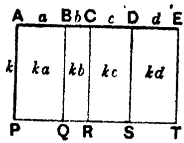


FIG. 415.

The whole figure APTE = the sum of APQB, BQRC, CRSD, DSTE;

∴ AP . AE = AP . AB + BQ . BC + CR . CD + DS . DE,

that is, $\quad k(a+b+c+d) = ka + kb + kc + kd.$

This can be enunciated thus:

If there are two str. lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the second line and each portion of the divided line.

FORMULA II.

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Let the line AC be divided at B into 2 parts of lengths a and b.

On AC describe a square ACRP; draw BQ ∥ AP.

Cut off AH $= b$, and draw HL ∥ AC.

It is easily seen that BL, AR and HQ are squares.

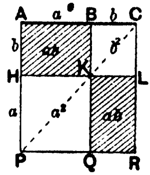


FIG. 416.

APRC = AHKB + BKLC + HPQK + KQRL;

$\therefore\ (a+b)^2 = ab + b^2 + a^2 + ab = a^2 + 2ab + b^2.$

This may be enunciated thus:

If a str. line is divided into any two parts, the square on the whole line is equal to the sum of the squares on each part, together with twice the rectangle contained by the parts.

It should be noticed that in this figure PKC is a straight line since $\hat{PKH} + \hat{HKB} + \hat{BKC} = 45° + 90° + 45° = 180°$.

FORMULA III.

$$(a-b)^2 = a^2 - 2ab + b^2.$$

Let AC $= a$, BC $= b$,

$\therefore$ AB $= a - b$.

Notice that BL, HQ and AR are squares.

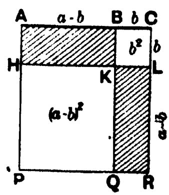


FIG. 417.

$(a-b)^2$

= HPQK = APRC − AHLC − KQRL

= APRC − AHLC − (BQRC − BKLC)

$= a^2 - ab - (ab - b^2)$

$= a^2 - 2ab + b^2.$

This may be enunciated thus:

The square on the difference between two lines equals the sum of the squares on the two lines less twice the rectangle contained by the lines.

FORMULA IV.

$$a^2 - b^2 = (a + b)(a - b).$$

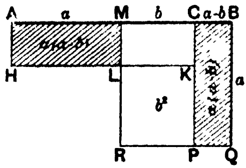


FIG. 418

Take $AB = 2a$; M the middle point of AB ;

$$MC = b \text{ ; } AH = BC = a - b \text{ ; } BQ = a \text{ :}$$

draw parallels as in the figure.

$$\begin{aligned} a^2 - b^2 &= MRQB - LRPK \\ &= MLKC + CPQB \\ &= MLKC + AHLM \\ &= AHKC \\ &= AC \cdot AH \\ &= (a + b)(a - b). \end{aligned}$$

This may be enunciated thus :

The difference of the squares on two str. lines is equal to the rectangle contained by their sum and difference.

Ex. *If* P, Q, R, S *are four points in order along a straight line, prove that*

$$PR^2 + QS^2 = 2QR \cdot PS + PQ^2 + RS^2.$$

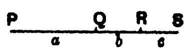


FIG. 419.

$$PR^2 = (a + b)^2 = a^2 + 2ab + b^2,$$

$$QS^2 = (b + c)^2 = b^2 + 2bc + c^2 ;$$

$$\therefore PR^2 + QS^2 = 2ab + 2b^2 + 2bc + a^2 + c^2,$$

$$2QR \cdot PS = 2b(a + b + c) = 2ab + 2b^2 + 2bc ;$$

$$\therefore 2QR \cdot PS + PQ^2 + RS^2 = 2ab + 2b^2 + 2bc + a^2 + c^2 ;$$

$$\therefore \text{ etc.}$$

EXERCISE 39.

1. Taking $a=1\cdot7$ ins., $b=1\cdot3$ ins., and $x=2$ ins., illustrate by a diagram the truth of the algebraic theorem $(a+b)x=ax+bx$.

2. Taking $a=1\cdot2$ ins., $b=1\cdot5$ ins., $x=0\cdot8$ ins., $y=1\cdot3$ ins., illustrate by a diagram that $(a+b)(x+y)=ax+bx+ay+by$.

3. Illustrate by a diagram that $a(b-c)=ab-ac$.

4. If A, B, C, D are four points in order on a straight line, prove that AC . BD = AD . BC + AB . CD.

5. If a straight line is 6 inches long and be divided into two parts, and the difference of the squares on these two parts is 8 sq. inches, calculate the lengths of the two parts.

6. A straight line is 7 inches long, and is divided into two parts so that the sum of the squares on the two parts is 29 sq. ins.; calculate the lengths of the two parts

7. If a straight line AB is divided into any two parts at C, prove that AC^2+CB^2 cannot be less than 2AC . CB.

8. If X is the middle point of a line PQ, and Y is a point in PQ produced, so that $XY^2=PQ^2+XQ^2$, prove that PY . QY $=PQ^2$.

9. ABC is an isosceles triangle with AB = AC, and BD is perp. to AC; prove that $BD^2=CD^2+2AD . DC$.

10. If a straight line AB is divided into two parts at C, prove that

$$4AB \cdot BC+AC^2=(2BC+AC)^2.$$

11. If a straight line AB is bisected at C, and divided *internally* unequally at D, prove that $AD^2+DB^2=2AC^2+2CD^2$ (Fig. 420).

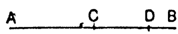


FIG. 420.

12. If a straight line AB is bisected at C and divided *externally* at D, prove that $AD^2+DB^2=2AC^2+2CD^2$ (Fig. 421).

FIG. 421.

13. If a straight line AB is divided into two parts at C, for what position of C will the value of AC^2+CB^2 be least?

14. Divide a straight line into two parts, such that the rectangle contained by those parts shall be the greatest possible.

15. AB is divided into two parts at C, and D, E are the middle points of AC and CB respectively. Show that the square on AE together with three times the square on EB is equal to the square on BD together with three times the square on DA.

16. If the line AB is bisected at C and produced to D, prove that

$$AD^2=BD^2+4AC \cdot CD.$$

NOTE ON PROJECTION.

The **projection of a point** on a straight line is the foot of the perpendicular drawn from the point to the line or the line produced.

The **projection of a straight line** on another (in the same plane) is that part of the latter which lies between the projections of the extremities of the former.

In the figure, XY is the line on which the points are projected.

H is the projection of A,
K ,, ,, B,
L ,, ,, C,
HK ,, ,, AB,
KL ,, ,, BC,
LH ,, ,, CA

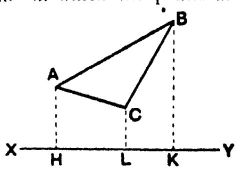


FIG. 422

THEOREM 40.

(An extension of Theorem of Pythagoras.)

In an obtuse-angled triangle, the square on the side opposite the obtuse angle is equal to the sum of the squares on the other two sides *together* **with twice the rectangle contained by one of these sides and the projection on it of the other.**

Let ABC be the triangle and A the obtuse angle.

BN is drawn perp. to CA produced.

It is required to prove that

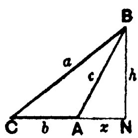


FIG. 423.

$$BC^2 = CA^2 + AB^2 + 2CA \cdot AN$$

Proof. Let $BC = a$, $CA = b$, $AB = c$, $AN = x$, $BN = h$.

$$BC^2 = CN^2 + NB^2 \qquad \text{(Pythagoras)}$$
$$= (b + x)^2 + h^2$$
$$= b^2 + 2bx + (x^2 + h^2)$$
$$= b^2 + 2bx + c^2 \qquad \text{(Pythagoras)}$$

i.e. $BC^2 = CA^2 + AB^2 + 2CA \cdot AN$,

or $a^2 > b^2 + c^2$ by $2bx$.

THEOREM 41.

(An extension of Theorem of Pythagoras.)

In any triangle, the square on the side opposite an acute angle is equal to the sum of the squares on the other two sides *diminished* by twice the rectangle contained by one of those sides and the projection on it of the other.

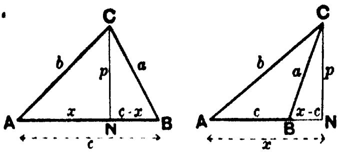


FIG. 424.

Let ABC be the triangle, A being the acute angle in each case.

CN is drawn perp. to AB or AB produced.

It is required to prove that

$$BC^2 = AB^2 + CA^2 - 2AB \cdot AN.$$

Proof. Let $BC = a$, $CA = b$, $AB = c$, $AN = x$, $CN = p$.

$$\begin{aligned} BC^2 &= BN^2 + CN^2 && \text{(Pythagoras)} \\ &= (c \sim x)^2 + p^2 \\ &= c^2 - 2cx + (x^2 + p^2) \\ &= c^2 - 2cx + b^2, && \text{(Pythagoras)} \end{aligned}$$

i.e. $BC^2 = AB^2 + CA^2 - 2AB \cdot AN$,

or $a^2 < b^2 + c^2$ by $2cx$.

THEOREM 42.

(Theorem of Apollonius.)

In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.

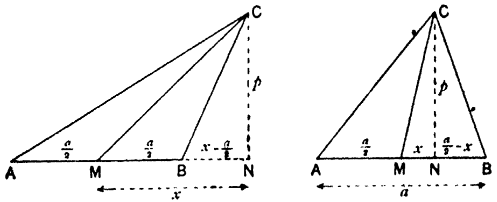


Fig. 425.

Let ABC be the triangle, M the middle point of the third side AB, and $\widehat{AMC}$ obtuse,

Draw CN perp. to AB produced or AB. Join CM.

It is required to prove that

$$CA^2 + CB^2 = 2AM^2 + 2CM^2.$$

Proof. Let $AB = a$, $MN = x$, $CN = p$;

$$\therefore \ NB = \frac{a}{2} \sim x.$$

$$AC^2 = AN^2 + NC^2 = \left(\frac{a}{2} + x\right)^2 + p^2 = \frac{a^2}{4} + ax + x^2 + p^2,$$

$$CB^2 = NB^2 + NC^2 = \left(\frac{a}{2} \sim x\right)^2 + p^2 = \frac{a^2}{4} - ax + x^2 + p^2;$$

$$\therefore \ AC^2 + CB^2 = \frac{a^2}{2} + 2(x^2 + p^2)$$

$$= 2AM^2 + 2CM^2.$$

EXERCISE 40.

1. Explain how to test whether a triangle is obtuse, right or acute-angled, when given the sides Apply your test to triangles whose sides are (*a*) 6, 8, 9, (*b*) 5, 6, 10, (*c*) 5, 12, 13.

2. State whether the triangles with the following sides are right-angled, acute angled, or obtuse angled, and give reasons :

(i) 2, 5, 6, (ii) 3, 4, 5, (iii) 3, 6, 7 ; (iv) 3, 7, 8.

3. If a line AB of length 8 cms makes an angle θ with another line XY, find by measurement the length of the projection of AB on XY when $\theta = 10°, 20°, 30°, 40°$ respectively.

4. If the sides of a triangle are 5, 6, 8 cms respectively, calculate the lengths of the medians to 1 place of decimals

5. If two sides AB and AC of a triangle ABC are 6 and 8 cms. respectively, and the median AD is 5·2 cms, calculate the length of BC.

6. If the medians of a triangle are 4, 5 and 6 cms., calculate the lengths of the sides to 1 place of decimals.

7. Show that the sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on its four sides.

8. If two sides of a parallelogram are 15·4 cms and 13·2 cms. and one diagonal is 16·7 cms, find the length of the other diagonal.

9. A point P moves so that the sum of the squares of its distances from two fixed points A and B is constant Prove that the locus of P is a circle

(Join P to M the middle point of AB, and prove that MP is constant.)

10. ABC is a triangle and M the middle point of BC. If AD is perp. to BC, prove that $AB^2 - AC^2 = 2BC . MD$.

11. Find the locus of a point which moves so that the difference of the squares of its distances from two fixed points is constant.

12. H is any point in the base BC of an isosceles triangle ABC. Prove that $AB^2 - AH^2 = BH . HC$.

13. ABC is an equilateral triangle, BC is trisected at H, K. Prove that $AH^2 = \frac{7}{9}AB^2$.

14. ACB is a triangle in which C is a right angle, and CD is the perpendicular on AB Given that $AB . BD = BC^2$, prove that

$$AB . AD = AC^2.$$

15. ABC is an isosceles triangle having AB = AC, and D is the foot of the perpendicular from C on AB Prove that $BC^2 = 2AB . BD$.

16. If G is the centroid of a triangle ABC, prove that

$$AB^2 + BC^2 + CA^2 = 3(AG^2 + BG^2 + CG^2).$$

17. In a triangle ABC, in which the angles at B and C are acute, BE and CF are drawn perp. to AC and AB respectively. Prove that

$$BC^2 = AB \cdot BF + AC \cdot CE.$$

18. In a quadrilateral ABCD, M and N are the middle points of AC, BD, prove that $AB^2+BC^2+AD^2+CD^2=AC^2+BD^2+4MN^2$ (Apply Apollonius' Theorem to the △s ABC, ACD, BMD.)

19. If ABC be an isosceles triangle, and the base BC be produced to H, and in AH a point E be taken, such that $AH \cdot AE = AC^2$, prove that $AH \cdot EH = BH \cdot HC$.

20. ABC is an isosceles triangle having AB = AC. DE is drawn parallel to BC to cut AB at D and AC at E. Prove that the difference between the squares on CD and CE is equal to the rectangle contained by BC and DE.

21. In an isosceles triangle ABC, the base BC is produced to D so that CD = BC. Prove that $AD^2 = AC^2+2BC^2$.

22. The sides AB, AC of a triangle ABC are equal; produce BC to D, and join AD. Show that the square on AD equals the square on AB, together with the rectangle BD . DC.

23. AB and CD are two parallel straight lines. AD is produced to E so that AD = DE, and BC is produced to F so that BC = CF. Prove that EF is parallel to AB and CD.

24. ABCD is a trapezium with AB and DC parallel sides. Prove that $AC^2+BD^2=AD^2+BC^2+2AB \cdot CD$.

25. ABC is a triangle in which the angle ABC = 60°; AN is perpendicular to BC.

Express AN in terms of AB, and CN in terms of BC and AB, and deduce that
$$AC^2 = BC^2 - BC \cdot AB + AB^2$$

ABCD is a regular tetrahedron, each edge of which measures 12 cms. The shaded triangle represents a section through C, E, F. If AE = 4 cms., determine the length of AF that will make the angle FEC a right angle. (*Army*.)

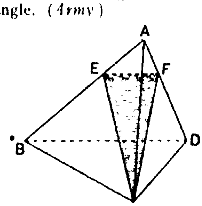


Fig. 126.

Note. Remember that the angles EAF, EAC and FAC are each 60°, and express CE^2, EF^2, FC^2 in terms of AC, AE, AF.

INTERSECTING CHORDS.

THEOREM 43.

If two chords of a circle intersect at a point within the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

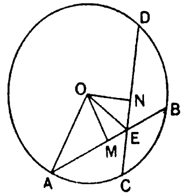


FIG. 427.

Let AB, CD be two chords of a circle intersecting at E; we have to prove that rect. AE . EB = rect. CE . ED.

Construction. Take the centre O and draw OM, ON perp. to AB, CD respectively, bisecting them at M and N.

Join OA, OE.

Proof.

$$\begin{aligned} AE \cdot EB &= (AM + ME)(MB - ME) \\ &= (AM + ME)(AM - ME) \\ &= AM^2 - ME^2 \\ &= (AM^2 + OM^2) - (ME^2 + OM^2) \\ &= OA^2 - OE^2. \qquad \text{(Pythagoras)} \end{aligned}$$

Similarly,

$$\begin{aligned} CE \cdot ED &= OC^2 - OE^2 \\ &= OA^2 - OE^2. \end{aligned}$$

$$\therefore\ AE \cdot EB = CE \cdot ED.$$

The *converse* is also true, and if two lines AB, CD intersect at E, so that AE . EB = CE . ED, then the four points A, B, C, D are concyclic.

N.B.—If r is the radius of the circle, note that we have proved that

$$AE \cdot EB = r^2 - OE^2.$$

THEOREM 44.

If from a point without a circle a secant and a tangent to the circle are drawn, the rectangle contained by the whole secant and the segment of it without the circle is equal to the square on the tangent.

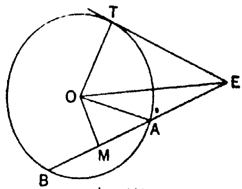


FIG. 428.

Let ET be a tangent to the circle and EAB a secant; we have to prove that EA . EB $=$ ET^2.

Construction. Take the centre O and draw OM perp. to AB, bisecting it at M. Join OT, OE, OA.

Proof.

$$\begin{aligned} EA \cdot EB &= (EM - MA)(EM + MB) \\ &= (EM - MA)(EM + MA) \\ &= EM^2 - MA^2 \\ &= (EM^2 + OM^2) - (MA^2 + OM^2) \\ &= OE^2 - OA^2 \qquad \text{(Pythagoras)} \\ &= OE^2 - OT^2 \\ &= ET^2. \qquad \text{(OT perp. to TE)} \end{aligned}$$

N.B.—If r is the radius of the circle,

$$EA \cdot EB = OE^2 - r^2.$$

COR. If two chords of a circle meet when produced at a point outside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

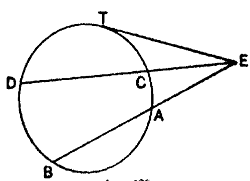


FIG. 429.

For

EA . EB $=$ ET^2 $=$ EC . ED.

The *converse* of this theorem is also true, and if two lines ET and EAB intersect at E so that ET^2 $=$ EA . EB, then the circle circumscribing the $\triangle$ABT touches ET at T.

The distance of the visible horizon.

If P is a point above the earth's surface and tangents be drawn from P to the earth, they will touch the earth in a circle, represented by a dotted line in the figure.

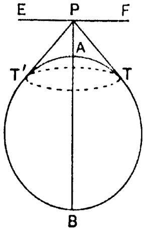


Fig. 130

This dotted circle is called the *Visible Horizon*.

PAB is drawn through the centre of the circle.

If a horizontal plane EPF (*i.e.* a plane parallel to the tangent plane at A) be drawn through the observer at P, then any one of the angles FPT or EPT′ is called the *Dip of the Horizon* and PT (= AT approx.), the *Distance of the Horizon*.

If r is the radius of the earth and PA $= h$, then since PT is a tangent and PAB a secant,

$$PT^2 = PA \cdot PB = h(h + 2r).$$

Since h will always be small in comparison with r, h^2 is smaller still, and may be neglected,

$$\therefore\ PT^2 = 2hr \text{ (approx.)}.$$

If now h is measured in *feet*, and r is taken as 3960 *miles*,

then
$$PT^2 = 2 \times \frac{h}{5280} \times 3960$$
$$= \tfrac{3}{2}h,$$
$$\therefore\ PT = \sqrt{\tfrac{3}{2}h} \text{ miles}.$$

Ex. *Find the distance of the horizon from the top of the cliffs of Moher, which are 600 ft. high.*

$$\text{Distance} = \sqrt{\tfrac{3}{2} \times 600} \text{ miles}$$
$$= \sqrt{900} = 30 \text{ miles}.$$

EXERCISE 41.

1. Through a fixed point P, 2·5 cms from O the centre of a circle of radius 6·5 cms., *any* chord BPC is drawn; find the constant numerical value of the area of the rectangle BP . PC for all positions of BC. In particular, if BP is equal to twice PC, calculate the length of BC.

2. Two chords of a circle intersect inside the circle; the segments of one chord are 2 and 7, and one of the segments of the other is 4; find the other segment.

3. A point is taken 6 inches from the centre of a circle, radius 2 inches. Calculate the length of the tangents from the point to the circle.

4. Draw a circle, radius 4 cms, and put in it a chord 6 cms long; produce the chord by 3 cms, and draw a tangent to the circle from the extremity of chord produced. Hence show how to find $\sqrt{27}$ by drawing.

5. OXY is a secant of a circle with centre A. If OX=3 cms, XY=5 cms and OA=6 cms, calculate the radius.

6. AOB is the diameter of a circle, centre O, and PNQ is a chord perp. to this diameter between O and A. If AO=2 ins, ON=$\frac{1}{2}$ in., calculate the value of BQ.

7. The radius of a circular arc ABC is 30 ft, and the height BD is 8 ft; find the span AC (Fig. 431).

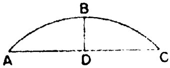


FIG. 431

8. O is the centre of a circle of radius 6·5 cms, and P is a point 5·5 cms. distant from O. A chord AB is drawn through P so that BP=3AP. Find the length of the chord.

9. Draw two lines AB and AC at an angle of 60°. In AC take P, Q such that AP=2, AQ=8. Take R in AB such that AR=4. Show that the circle PQR touches AB at R, and explain why. Draw any other circle through P, Q cutting AB at L, M. Compare the length of the tangent from A to this circle with the length of AR.

10. AT is a tangent and AD a diameter of a circle; TB is parallel to AD and BC perp. to AD (Fig. 432).

If AC=x and the radius=a, prove $TA^2=x(2a-x)$.

(This is often useful in Mechanics when x is small, and consequently $TA^2=2ax$, approx.)

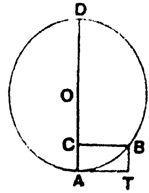


FIG. 432.

11. ACB is a chord of a circle such that AC=4 cms., CB=5¼ cms., OC=2 cms., where O is the centre of the circle. Calculate the radius of the circle.

12. Find the distance of the visible horizon for an aeroplane 8000 ft. above sea level. (Radius of earth = 3960 miles.)

13. Find the distance of the visible horizon from the top of a mast 120 ft. above sea-level. (Radius of earth = 3960 miles.)

14. Given the distances above sea-level, find the distance of the visible horizon from

(i) Beachy Head, 575 ft.,
(ii) Holyhead Mountain, 719 ft.,
(iii) Lands' End Hotel, 216 ft.,
(iv) Telegraph Inn, Great Orme's Head, 679 ft.,
(v) The Rivals (Carnarvonshire), 1849 ft.,
(vi) Snowdon, 3560 ft.

In each case assume that the earth's radius is 3960 miles.

15. From the top of one lighthouse 140 ft high the light of another 180 ft. high can just be seen Calculate the approximate distance apart of the lighthouses, if the radius of the earth is taken as 3960 miles.

16. Two circles intersect at A and B From a point in BA produced are drawn two straight lines, one cutting one of the circles in K and L, and the other cutting the other circle in M and N. Prove that the points K, L, M, N are concyclic (Fig. 433).

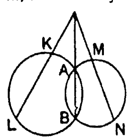


FIG. 433.

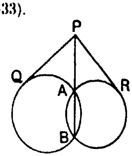


FIG. 434.

17. Two circles intersect in A and B, and from any point P, on BA produced, tangents PQ, PR are drawn to the circles. Prove PQ = PR (Fig. 434).

18. Two circles intersect in A and B, and BA produced meets a common tangent ST in X. Prove XS = XT (Fig. 435).

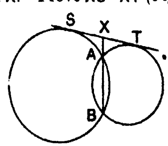


FIG. 435.

19. A is a point on the common chord of two circles. LAM is a chord of one circle and PAQ of the other. Prove that a circle can be drawn through L, M, P, Q.

20. ABC is a triangle; AD is perpendicular to BC; DE, DF are perpendicular to AC and AB respectively. Prove that (i) AD touches the semi-circles DEC and BDF; (ii) AE . AC = AD^2; (iii) BFEC is cyclic.

21. If PA, PB are two tangents to a circle, centre O, from an external point, and OP meets AB in Q, prove that OQ . OP = OA^2.
(Prove that OA is a tangent to the circle through A, Q, P.)

22. The diagonals of a cyclic quadrilateral ABCD meet at X and DX = BX. Prove that the sum of the squares on the sides of the quadrilateral equals $2AC^2$.

23. If three circles intersect, the three common chords of the circles, taken in pairs, are concurrent [Hint: call the circles P, Q, R; let the common chords of P, Q, and Q, R intersect at O, and

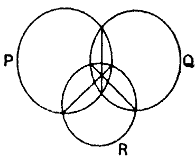


Fig. 436.

let AB be the common chord of P, R. Join AO, and suppose it meets P at K and R at K'; then prove OK = OK', and hence K and K' must both lie at B.] (Fig. 436.)

24. A semi-circle is drawn with AB as diameter, and two chords AX, BY are drawn intersecting at C; prove that

$$AC \,.\, AX + BC \,.\, BY = AB^2$$

(Draw CM perp. to AB, and note that $\hat{X} = \hat{Y} = 90° = C\hat{M}B$.)

25. Given two fixed circles, centres O and O', let P be a movable point such that the two tangents PA, PB are equal

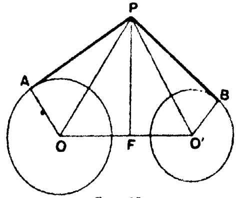


Fig. 437.

Prove that the locus of P is a line perp. to OO'. This locus is called the *radical axis* of the two circles.
(Prove $OF^2 - O'F^2$ is constant, where PF is perp. to OO'.)

26. From a point A outside a circle a secant ABC and a tangent AD are drawn so that AC is perpendicular to AD. Prove that $AB^2+AC^2+2AD^2$ is four times the square of the radius of the circle.

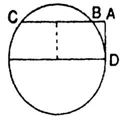


FIG. 138.

27. From P, Q the ends of a diameter of a circle, perpendiculars PA, QB are drawn to a chord XY of the circle; show that the rectangle AX . XB is equal to PA . QB

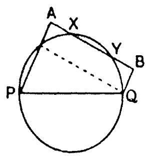


FIG. 139.

*28. ABCD is a rectangle inscribed in a circle. From a point P in the circumference of the circle perpendiculars PH, PK, PL, PM are drawn to the four sides AB, BC, CD, DA Show that PH . PL = PM . PK.

29. PQRS is a square O is the middle point of RS. The circle POQ cuts PS again at T Prove that PT = 3TS

30. A system of circles is drawn through two points A, B, and P is a given point on AB produced. Find the locus of the points of contact of the tangents drawn from P to the circles.

31. ABC is a triangle, a circle passes through A and touches BC at D, and it is found AB = 3BD; if the circle cuts AB at K, prove AK = 8BK.

32. ABC is an isosceles triangle with BA = BC. A circle passing through A and touching BC at its middle point cuts AB in X; prove that AX = 3BX.

33. If two circles touch externally, prove that the square on their common tangent equals the rectangle contained by the two diameters.

CONSTRUCTION 14.

Describe a square equal in area to a given rectangle.

Let ABCD be the given rectangle.

Produce AB to E, so that BE = BC.

Bisect AE at O, and with O for centre and radius OA describe a semicircle.

Produce CB to meet the semicircle on F; then the square described on BF is equal in area to the given rectangle.

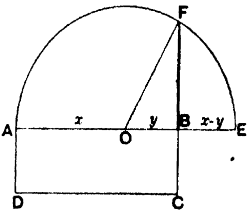


Fig. 440

Method i. Join OF.

Let AO = OE = x and OB = y.

$$\begin{aligned} FB^2 &= OF^2 - OB^2 \\ &= OE^2 - OB^2 \\ &= x^2 - y^2 = (x+y)(x-y) \\ &= AB \,.\, BE = AB \,.\, BC \\ &= \text{given rectangle.} \end{aligned}$$

Method ii. Since ABE is a diameter, the chord FB produced is bisected at B, since it is perp. to ABE;

$$\therefore\ FB^2 = AB \,.\, BE \qquad \text{(Th. 43)}$$
$$= AB \,.\, BC.$$

Method iii. Let ABCD be the rectangle.

Cut off BE = BC; on AB as diameter describe a semicircle.

Draw EF perp. to AB. Join BF.

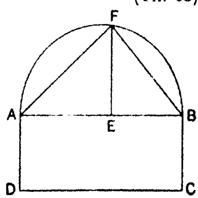


Fig. 441.

$$\begin{aligned} \text{Rect. } ABCD &= AB \,.\, BE \\ &= (AE + EB)\, EB \\ &= AE \,.\, EB + EB^2 \\ &= EF^2 + EB^2 && \text{(Th. 43)} \\ &= BF^2. && \text{(Pythagoras)} \end{aligned}$$

It will be noticed that in Method iii. the figure is drawn so that the two lines AB, BE overlap, whereas in Methods i. and ii., BE is a continuation of AB, and consequently the figure takes up more space.

CONSTRUCTION 15.

Describe a square equal in area to a given polygon.

This problem involves describing a triangle equal in area to the given polygon, reducing the triangle to a rectangle, and then drawing a square equal in area to the rectangle.

To draw a circle through two given points so as to touch a given line.

Let the line joining the two given points A, B meet the given line XY at T.

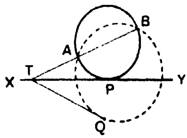


FIG. 442.

Take TP such that $TP^2 = TA \cdot TB$.

The circum-circle of PAB touches XY at P.

The point P may be conveniently found geometrically by drawing any circle through A, B, drawing a tangent TQ to this circle, and then making TP = TQ.

To draw a circle through two given points to touch a given circle.

Let A, B be the given pts., and PQX the given circle.

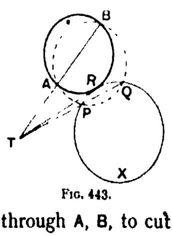


FIG. 443.

Draw any circle through A, B, to cut the given circle at P, Q.

Let BA and QP meet at T.

Draw TR a tangent to the given ⊙.

Then $TA \cdot TB = TP \cdot TQ = TR^2$;

$\therefore$ the circum-circle of ABR will touch the line TR and the given circle at R. (Th. 44)

EXERCISE 42.

1. Describe a rectangle, with one side 7·5 cms. long, equal in area to a square of side 5 cms. Measure the other side of the rectangle.

(Draw AB = 5 cms. and AC = 7·5 cms., so that $B\hat{A}C = 90°$. Draw BD perp. to BC to meet CA produced at D.)

2. If one side of a rectangle is 5·2 cms., find by construction the other side, if the area of the rectangle is 16 sq. cms.

3. Show how to divide a given straight line into two parts such that the rectangle contained by them shall be as great as possible.

4. Draw a straight line 9·5 cms. long, and by geometrical construction divide it into two parts so that the rectangle contained by these parts shall be 16 sq. cms. in area

5. Construct a square having its area equal to that of an equilateral triangle whose sides are 5 cms. Measure the side of the square and check the result by calculation

6. Construct a triangle with sides 4 ins., 3·4 ins., 2·6 ins. Also construct a square equal in area to the triangle. Measure a side of the square.

7. Construct a square of side 2 inches, and then construct a rhombus of side 2·5 inches of equal area. Measure its diagonals.

(First reduce square to a rectangle with one side 2·5 ins.)

8. Construct a square equal in area to the quadrilateral ABCD in which AB = 3 ins, BC = 2·7 ins, CD = 2·1 ins., AD = 4 ins, and the angle ABC = 115°. Measure a side of the square.

9. Inscribe a regular hexagon in a circle of radius 2 inches.

Draw a square whose area is equal to that of the hexagon, and measure the side of the square

10. Draw a regular pentagon with each side 1·5 inches, and then describe a square equal in area Measure the side of the square.

(Each angle of the pentagon is 108°.)

11. A flagstaff AB, 20 ft. high, stands at the top of a tower BC, of height 80 ft. Find a point D in the horizontal plane through C at which the flagstaff subtends a maximum angle.

(Draw a ⊙ passing through A, B and touching the line through C perp. to AC.)

12. Make an angle AOB = 30°. In OA take points P, Q such that OP = 4 cms., OQ = 9 cms. Describe a circle passing through P, Q, and touching OB. Measure the radius.

13. Draw two straight lines, OA, OB inclined at an angle of 50°; on OA take two points P, Q such that OP = 4·4 cms. and OQ = 9·6 cms.; through P and Q draw a circle to touch OB. Measure the radius.

14. Draw a circle, centre O, radius 3 cms. Draw any line OA; through O draw OB, OC so that $A\hat{O}B = 15°$, $B\hat{O}C = 25°$. On OB, OC take points X and Y, so that OX = 5 cms., OY = 5·5 cms. Draw a circle to pass through X and Y and touch the given circle. Measure its radius.

RATIO AND PROPORTION.

In Fig. 444 the length of the straight line AB contains the length of the shorter line PQ exactly 3 times ;

AB is said to be a *multiple* of PQ,

and PQ ,, ,, *submultiple* of AB.

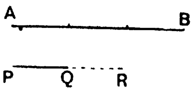


FIG. 444

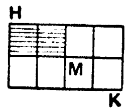


FIG. 445

In Fig. 445 HK contains the small (shaded) area HM exactly 4 times ;

the area HK is a multiple of fig. HM,

and the area HM is a submultiple of fig. HK.

DEF. A greater magnitude which *exactly* contains a smaller magnitude a definite number of times is said to be a *multiple* of the smaller magnitude ; and the smaller magnitude is said to be a *submultiple* of the greater.

Let PQ (Fig. 444) be produced to R, so that PR = 2PQ.

AB is not a multiple of PR, but AB and PR have this relation between them, that each is a multiple of PQ.

PQ is sometimes called a *common measure* of AB and PR.

Similarly in Fig. 445, the shaded part HM is a *common measure* of the whole area HK and the unshaded part.

DEF. If two magnitudes are such that each is a *multiple* of some third magnitude, the two magnitudes are said to be **commensurable.**

The subject of *commensurable* and *incommensurable* magnitudes presents at first some difficulty.

It usually strikes beginners as curious that two straight lines can be *incommensurable.*

It is a fact, however, that no common measure can ever be found which is an exact submultiple of the side of a square and also its diagonal.

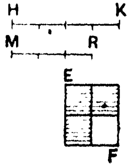
DEF. The **Ratio** of one quantity to another *of the same kind*, expressed in terms of the same unit, is the quotient of the one divided by the other.

The ratio of a to b may be written $\frac{a}{b}$, $a : b$, $a \div b$, a/b.

Proportion. DEF. Proportion is the equality of Ratios.

Thus if $a : b = c : d$, the four quantities a, b, c, d, form a proportion, d being called the *fourth* proportional.

$$\frac{\text{The length of line HK}}{\text{The length of line MR}} = \frac{4}{3},$$



also
$$\frac{\text{area of whole fig. EF}}{\text{area of shaded part}} = \frac{4}{3}.$$

Thus line HK, line MR, area EF, and shaded part of EF, constitute 4 magnitudes in proportion.

FIG. 446

Duplicate Ratio. The duplicate ratio of $a : b$ is $a^2 : b^2$.

Mean Proportional. If there are three magnitudes such that $1^{st} : 2^{nd} = 2^{nd} : 3^{rd}$, then the 2^{nd} magnitude is called the *mean proportional* to the 1^{st} and 3^{rd}.

Third Proportional. In the above case, the 3^{rd} magnitude is said to be the *third proportional* to the first two.

Thus, if $a : b = b : c$,

b is the *mean proportional* between a and c,

c is the *third proportional* to a and b.

The following algebraic results should be known.

If $a : b = c : d$, then

(i) $b : a = d : c$,

(ii) $a : c = b : d$,

(iii) $ad = bc$,

(iv) $\frac{a+b}{b} = \frac{c+d}{d}$,

(v) $\frac{a-b}{b} = \frac{c-d}{d}$,

(vi) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

DIVISION OF A LINE IN A GIVEN RATIO.

Internal Division. A straight line AB is said to be divided internally at P in the ratio $x : y$, if AB can be divided into $x+y$ equal parts of which AP contains exactly x and PB exactly y;

$$i.e.\ AP : PB = x : y.$$

FIG. 447.

External Division. A straight line AB is said to be divided externally at P in the ratio $x : y$ (x being greater than y), if

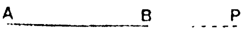


FIG. 448.

AB can be divided into $(x-y)$ equal parts, of such a length that if marked off successively along BP, BP contains exactly y of them, and consequently AP exactly x;

$$i.e.\ AP : BP = x : y.$$

If $y > x$, then AB must be divided into $(y-x)$ parts and P would be in BA produced.

Extreme and Mean Ratio. If a line AB is divided into two parts at X, so that

$$AB : BX = BX : AX,$$

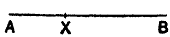


FIG. 449.

then it is said to be divided into extreme and mean ratio, or in *medial section*.

Note that **the areas of any two triangles which have equal altitudes are proportional to their bases; or if they have equal bases, they are proportional to their altitudes.**

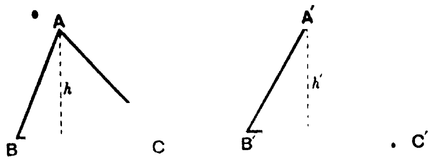


FIG. 450.

The area of any triangle = $\frac{1}{2}$ base × altitude.
If h and h' be the altitudes, we have

$$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\frac{1}{2}BC \times h}{\frac{1}{2}B'C' \times h'}$$

$$= \frac{BC}{B'C'}, \qquad \text{if } h = h'.$$

Also,

$$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\frac{1}{2}BC \times h}{\frac{1}{2}B'C' \times h'}$$

$$= \frac{h}{h'}, \qquad \text{if } BC = B'C'.$$

EXERCISE 43.

1. Find the ratio of

(i) 12 inches to 15 inches, (ii) 12 inches to 3 feet,

(iii) 10 sq. ins. to 35 sq. ins., (iv) 48 sq. ins. to 1 sq. ft.

2. Express the ratios (i) $\frac{15 \cdot 6}{0 \cdot 31}$, (ii) $\frac{8 \cdot 26}{17 \cdot 53}$, (iii) $\frac{0 \cdot 56}{8 \cdot 72}$, as decimals correct to 2 places.

3. Measure the lines A and B (i) in inches, (ii) in cms., (iii) in mms., and thus find the ratio of A to B. Express the answers in decimals.

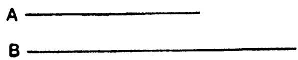


FIG. 451

4. Find, by measurement, the areas of the rectangles A and B in sq. ins. and sq. cms., and thus find in two ways the ratio of A to B.

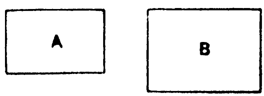


FIG. 452.

5. Find the values of x in the following proportions:

(i) $7 : 10 = 2 : x$,

(ii) $3 : x = 12 : 20$,

(iii) $4 : 5 = x : 15$

6. Find the values of x in the following proportions, and state in what unit x is measured:

(i) 3 ft. : 8 ft. = 15 lbs : x,

(ii) x : 9 ozs. = 8 ins. : 3 ins.,

(iii) 7 cu. ft. : x = 14 yds. : 10 yds.

7. Write in the form of a ratio (i) $3 \times 16 = 6 \times 8$, (ii) $ab = cd$.

8. A line 7·5 cms. in length is divided internally in the ratio 3 : 5; calculate the lengths of the segments.

9. A line 5·2 inches long is divided externally in the ratio 3 : 7; calculate the lengths of the segments.

10. Calculate (i) a third proportional to 3 and 8,

(ii) a mean proportional to 3 and 75.

THEOREM 45.

A straight line drawn parallel to one side of a triangle divides the other sides proportionally.

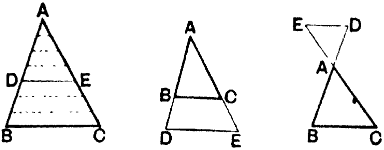


FIG 453.

Let ABC be a triangle and DE a line $\parallel$ BC. We have to prove that

$$AD : DB = AE : EC,$$

$$\text{or } \frac{AD}{DB} = \frac{AE}{EC}.$$

Proof. Suppose AD and DB are commensurable. Take any common measure of AD and DB as unit of length, and suppose AD contains m units of length and DB p units.

Divide AD into its m units and DB into its p units, and through each point of division draw parallels to BC.

Then the parallels divide AE into m equal parts all of a certain length, and EC into p equal parts, each of the same certain length.

$$\therefore \text{ AE} : \text{EC} = m : p,$$

$$\text{but AD} : \text{DB} = m : p;$$

$$\therefore \text{ AD} : \text{DB} = \text{AE} : \text{EC}.$$

THEOREM 46.

If two sides of a triangle are divided in the same ratio, the straight line joining the points of section is parallel to the third side.

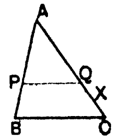


FIG. 454

Let ABC be a triangle having its sides cut by PQ so that

$$AP : PB = AQ : QC.$$

We have to prove that PQ is $\parallel$ BC.

Proof. Suppose PQ is not parallel to BC, and if possible let PX be parallel to BC.

Then $AP : PB = AX : XC$,

but $AP : PB = AQ : QC$; (Hypoth.)

$\therefore\ AX : XC = AQ : QC$,

i.e. AC is divided internally in the same ratio at two points, and this is impossible.

$\therefore$ PQ must be parallel to BC.

CONSTRUCTION 16.

To find a fourth proportional to three given straight lines.

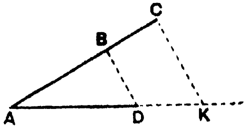


FIG. 455.

Let AB, BC, AD be the three given straight lines.

Construction. Place them as in the figure, with the first two in the same str. line.

Join BD, and draw CK || BD to meet AD produced at K.

Proof. Then AB : BC = AD : DK. (BD || CK)

Hence DK is the required Fourth Proportional.

If BC and AD are equal, then DK is the **Third Proportional** to AB, BC.

CONSTRUCTION 17.

To divide a given straight line, internally or externally in a given ratio, e.g. 3 : 2.

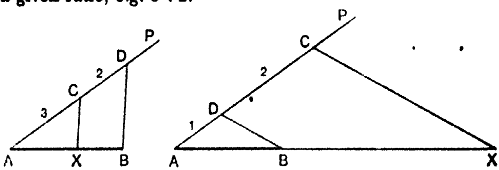


FIG. 456

Let AB be the given straight line which is to be divided internally or externally in the ratio 3 : 2.

Construction. Draw AP of indefinite length at any convenient angle to AB.

Cut off AC = 3 units and CD = 2 units, CD being a continuation of AC for internal division, and overlapping AC for external division.

Join DB and draw CX || DB, meeting AB, or AB produced at X.

Proof. Since CX || DB,

$$\therefore \ AX : XB = AC : CD = 3 : 2.$$

EXERCISE 44.

1. Draw a line 3·2 inches long, and divide it internally into two parts in the ratio of 3 to 5. Measure the parts.

2. Divide a line 4·7 inches long internally into two parts proportional to 2 and 7. Measure the parts.

3. Divide a line 6·2 cms long internally into two parts proportional to the lengths of A and B. Measure the two parts.

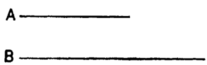


FIG. 457.

4. Divide a line 4·3 cms long externally in the ratio of the lengths of A and B, and measure the parts

5. Draw a line of length 6·7 cms, and divide it externally in the ratio 5 : 3. Measure the parts.

6. Draw a line 11·2 cms long, and show how to cut off one-seventh part

7. Draw a line 8·7 cms. long, and cut off two-fifths of it. Measure this part.

8. Divide a line AB, 4·7 cms. long, internally at P and externally at Q, in the ratio 2 . 5. Measure the parts and show that PQ is divided internally at A in the same ratio as it is divided externally at B. What is this ratio?

9. Draw a line 8·4 cms long, and divide it into 3 parts proportional to 2, 3, and 5. Measure the parts.

10. Divide a line 4·8 inches long into three parts proportional to 1, 3, and 5. Measure the parts.

11. Construct the fourth proportional to 3, 4 and 5, and measure it.

12. Find, by construction, the fourth proportional to 4, 5 and 6. Measure it.

13. Construct the third proportional to 3 and 4. Measure it.

14. Find, by construction, the third proportional to 4 and 5.

15. If the sides of a rectangle are 4·2 and 3·4 cms., and another rectangle equal to it in area has one side 2·7 cms., find its other side by construction.
(Find a 4th proportional to 2·7, 4·2, 3·4.)

16. ABC is a triangle and HK a line parallel to BC, meeting AB and AC produced at H and K; given AB=3 cms., HB=5 cms., AK=4 cms., calculate KC.

17. P is any point between the arms of an angle XOY. Show how to draw a line MPN terminated by OX and OY, such that

(i) MN is bisected at P,
(ii) MP : PN = 5 : 7.

18. Draw a line through the vertex A of a triangle ABC so as to divide the area of the triangle in the ratio 2 : 3.

19. If AX : XB = 3 : 1 and AY : YC = 4 : 1 (Fig. 458), prove that

$$\triangle ABY : \triangle ABC = 4 : 5,$$

and

$$\triangle AXY : \triangle ABY = 3 : 4.$$

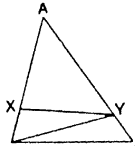


FIG. 458.

20. O is a fixed point, and a line OP (variable in direction) is drawn to cut a fixed line at P. Q is taken on OP such that the ratio OQ : QP is fixed; what is the locus of Q?

21. ABCD is a parallelogram and P a point on BD such that BP : PD = 3 : 5. Prove that $\triangle ABP = \frac{3}{16}$ of the parallelogram.

22. If XY and AB are two parallel lines, prove that

$$\triangle AXY : \triangle ABX = XY : AB$$

23. The side BC of an equilateral triangle ABC is produced to D so that CD = BC. Prove that the perpendiculars to AC, drawn through B and C respectively, trisect AD.

24. ABCD is a parallelogram; E and F are the middle points of AB and CD. Show that AC is trisected by DE and BF.

25. ABC is any triangle, and O is the centre of its circumscribing circle. From O is drawn a perpendicular to BC, cutting BC at D. From D is drawn DE parallel to BA, cutting AC at E. Prove that OE is perp. to AC.

26. ABC is a triangle and D the middle point of BC. A point E is taken in AC, such that CE : EA = 2 : 1. If AD and BE cut at P, prove that AP = PD.
(Draw DF ∥ to BE.)

27. Two circles ABP, ABQ intersect in A and B; C is the point midway between their centres. If PBQ be drawn at right angles to CB, show that PB and BQ will be equal.
(Draw perps. from the two centres to PBQ.)

THEOREM 47.

The bisector (internal or external) of an angle of a triangle divides the opposite side (internally or externally) in the ratio of the sides containing the angle bisected.

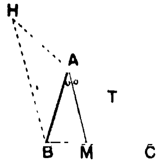


FIG. 459.

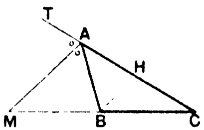


FIG. 460.

(i) Let ABC be a triangle, having AM drawn bisecting $\hat{A}$ internally or externally.

We have to prove

$$AB : AC = BM : MC.$$

Construction. Draw BH parallel to MA, meeting CA produced or CA at H.

In Fig. 459, let T be in AC, and Fig. 460, let T be in CA produced.

Proof. $\because$ MA and BH are parallel,

$$A\hat{H}B = T\hat{A}M, \quad \text{(corresponding angles)}$$

$$\text{and } A\hat{B}H = M\hat{A}B; \quad \text{(alt. angles)}$$

$$\therefore A\hat{H}B = A\hat{B}H;$$

$$\therefore AB = AH.$$

Now $$HA : AC = BM : MC, \quad (AM \parallel HB)$$

$$\therefore AB : AC = BM : MC.$$

The *converse* of this proposition is also true and can be proved with the same construction.

EXERCISE 45.

1. Draw a triangle ABC with $a=4$, $b=2$, $c=5$ cms.; bisect the internal and external angles at A by AX, AY, meeting BC and BC produced in X and Y. Measure BX, XC, BY, YC, and thence find the values of the ratios BX : XC, BY : YC.

2. In a $\triangle$ABC, if AK bisects the angle BAC, K is on BC such that BK=15 cms., KC=20 cms., and the angle A is a right angle, show that

$$AB+AC=49 \text{ cms.}$$

3. Inscribe a $\triangle$ in a circle of radius 2·3 cms. such that the two segments into which the bisector of the vertical angle divides the base are of lengths 1·5 cms and 1·9 cms. Measure the two sides of the triangles, and show that there are two solutions.

4. In a circle of radius 1·4 ins., inscribe a triangle, having its base 2·5 ins. long, and the ratio of its remaining sides 2 : 3. Measure the lengths of these sides.

5. The side BC of a triangle ABC is bisected at D; the angles ADB, ADC are bisected by straight lines meeting AB and AC at M and N. Prove that MN is parallel to BC.

6. ABCD is a quadrilateral having AB=AD. The angles BAC, CAD are bisected by lines meeting BC and CD at E and F. Prove that EF is parallel to BD.

7. A and B are two fixed points, and a point P moves so that the ratio PA : PB is fixed (Fig. 461).

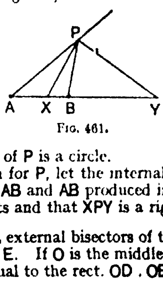


FIG. 461.

Prove that the locus of P is a circle.

(Taking any position for P, let the internal and external bisectors of the angle APB meet AB and AB produced in X and Y. Prove that X and Y are fixed points and that XPY is a right angle.)

8. The internal and external bisectors of the angle C of a triangle ABC meet AB in D and E. If O is the middle point of AB, show that the square on OB is equal to the rect. OD . OE.

9. A and B are two fixed points 5 cms. apart, and a point P moves so that PA : PB=5 : 3. Draw several positions of P and thus construct its locus. Measure the diameter of the circle obtained.

10. AB is a diameter of a circle. C is any point on a chord PQ of the circle perp. to AB. AC and BC cut the circle in R and S respectively. Prove that PR : RQ=PS : SQ.

SIMILAR FIGURES.

DEF. Two plane rectilinear figures are said to be **equiangular** if the angles of one *taken in order* are respectively equal to the angles of the other taken in order.

DEF. Plane rectilinear figures are said to be **similar** if (i) they are equiangular **and** (ii) the corresponding sides are in the same ratio.

In the case of *triangles* we shall prove that if one of these conditions holds then the other necessarily follows.

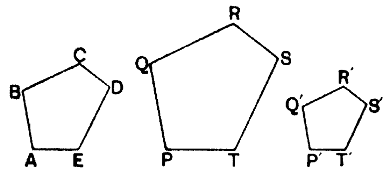


FIG. 462

Thus if $\hat{A} = \hat{P}$, $\hat{B} = \hat{Q}$, $\hat{C} = \hat{R}$, $\hat{D} = \hat{S}$, $\hat{E} = \hat{T}$,

and if $AB : PQ = BC : QR = CD : RS = DE : ST = EA : TP$,

then ABCDE, PQRST are said to be similar.

Without considering the subject from a definitely mathematical point of view, the student probably has a very fair idea of the meaning of similar figures from the consideration of a lantern slide and its projection on a screen, and knows that if corresponding lines are not proportional, and corresponding angles not equal, then the picture shown is not a correct representation of the slide, but is a distortion.

COR. Figures which are similar to the same figure are similar to each other.

For if P′Q′R′S′T′ be any other figure similar to ABCDE,

then $$\hat{P} = \hat{A} = \hat{P}',$$

$$\hat{Q} = \hat{B} = \hat{Q}', \text{ etc.,}$$

and $$PQ : QR = AB : BC = P'Q' : Q'R',$$

$$QR : RS = BC : CD = Q'R' : R'S', \text{ etc.}$$

THEOREM 48.

If two triangles are equiangular, their corresponding sides are proportional, *i.e.* the triangles are similar.

Let ABC and PQR be two triangles, having $\hat{A}=\hat{P}$, $\hat{B}=\hat{Q}$, and $\therefore \hat{C}=\hat{R}$.

We have to prove

$$\frac{AB}{PQ}=\frac{BC}{QR}=\frac{CA}{RP}.$$

Proof. Apply ABC to PQR so that $\hat{A}$ coincides with $\hat{P}$, and BC takes up the position B′C′.

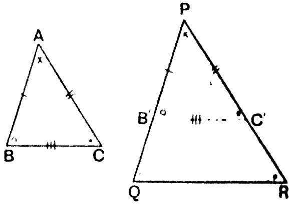


FIG. 463.

$$\because \; P\hat{B'}C' = A\hat{B}C = P\hat{Q}R,$$

$\therefore$ B′C′ is parallel to QR ; (corr. angles)

$$\therefore \frac{PB'}{PQ}=\frac{PC'}{PR}, \quad \text{(similar div.)}$$

$$\text{or} \quad \frac{AB}{PQ}=\frac{AC}{PR}.$$

Similarly, by making $\hat{B}$ coincide with $\hat{Q}$, it may be proved that

$$\frac{AB}{PQ}=\frac{BC}{QR}.$$

The student must note carefully that if two polygons are equiangular, it does not necessarily follow that they are similar.

If B′C′ is parallel to BC, then the quadrilaterals ABCD, AB′C′D are obviously equiangular ; if the quads. were similar, then

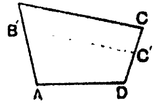


FIG. 464.

$$\frac{AB}{AD}=\frac{AB'}{AD},$$

i.e. AB = AB′,

an obviously impossible result.

EXERCISE 46.

1. Draw two triangles each having angles 40°, 60°, 80°, with the longest side 4 cms. in one and 7 cms. in the other. Measure the shortest sides, and verify that they are in the ratio 4 : 7.

2. Draw two triangles each having angles 30°, 100°, 50°, with the side opposite the 50° 4 cms. in one and 5 cms. in the other. Measure the longest sides, and verify that they are in the ratio 4 : 5.

3. A house casts a shadow 80 feet long and a stick 5 ft. high casts a shadow 4 ft. long; find the height of the house.

4. A man of 6 ft. is standing 6 ft. from a lamp-post, and his shadow is 9 ft.; find the height of the light above the ground.

5. ABCD is a trapezium in which AB = 1·5 cms., CD = 3·2 cms., and the distance between AB and DC = 2 cms. If DA and CB produced meet at V, calculate the perp. distance of V from DC (Fig. 465).

6. In Fig. 465, if VE = 2 cms., EF = 3 cms., and DC = 2·9 cms., calculate the length of AB.

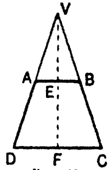


FIG. 465.

7. To calculate the breadth AB of a river, a distance BD along the bank is measured and found to be 72 yards, DE (at right angles

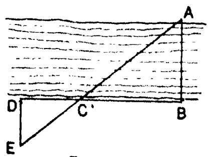


FIG. 466.

to BD) is 9 yards; a post is placed at C, in BD, so that E, C and A are in the same straight line. On measurement, DC is found to be 22 yards. Calculate the breadth AB, where AB is perp. to BD.

8. B is a point visible but inaccessible from another point A, from which it is required to find its distance. A point C is taken 135 yards from A, and such that CA is at right angles to AB. Along BA produced E is taken, 80 yards from A. From E a line is drawn at right angles to EB, and a point F is taken on it such that F, C, B are in a straight line. It is found that EF is 150 yards long. Calculate the distance AB.

9. Find, to one place of decimals, the height of a cone whose slant side is 10 cms. and base-radius 3 cms. If the cone is inserted, vertex downwards, into a horizontal circular hole of radius 2 cms., how deep will the vertex be below the plane of the hole?

10. The Fig., which is not drawn to scale, shows the section of a roof and the beams supporting it. The span AB is 25 feet and is

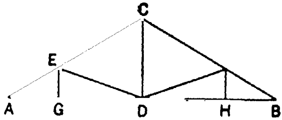


FIG. 467.

horizontal; the vertical post DC is 8 feet and rests on the middle point D of AB. A strut ED connects D to a point E of AC, which is 6 ft. 6 ins. from A. A vertical post GE completes the left half of the section.

Calculate the length of GE and the distance GD. Hence obtain the length of the strut DE. Verify your calculations by an accurate drawing made to scale. (*Army*)

11. AOB, COD are any two chords of a circle; prove that the triangles AOD, BOC are similar, and hence that AO . OB = CO . OD.

12. ABC is a triangle, right-angled at A, and AD is drawn perpendicular to BC; prove that the triangles BDA and ADC are similar, and hence BD . DC $=$ AD^2.

13. From an external point P a tangent PT is drawn to a circle, and also a secant PAB; prove that the triangles PTA and PBT are similar, and hence PA . PB $=$ PT^2.

14. ABCD is a cyclic quadrilateral and AB and DC meet when produced at O; prove that the triangles OAD and OCB are similar, and hence OA . OB = OD . OC.

15. ABC is a triangle, and the bisector of A meets BC at D and the circumcircle at E; prove that the triangles BDA, ECA are similar, and hence AB . AC = AE . AD.

16. X, Y are points on the sides AB, AC of a triangle ABC such that XY is parallel to BC. BY, CX meet at O, and a line through B parallel to CA meets CX produced in Z. Prove that OX . OZ $=$ OC^2.

17. ABCD is a cyclic quadrilateral, E is a point on BD such that $B\hat{A}E = C\hat{A}D$; prove that the triangles ABE and ACD are similar, and hence BA . CD = BE . AC. (See Qn. 30.)

18. PQR, PQS are two triangles on opposite sides of the common base PQ. RS meets PQ, or PQ produced, at T. Prove that

$$\triangle PQR : \triangle PQS = RT : TS.$$

19. Two given circles touch each other internally at a point O, and from O a line OPQ is drawn, meeting the circles in P and Q. Prove that the ratio OP : OQ is constant for all positions of this line.

20. OA is a fixed chord of a circle, and OB a variable chord. The line through A parallel to the tangent at O cuts OB in C. Prove that the rectangle OB . OC is constant.

21. The angle C of a triangle ABC is a right angle and CD is drawn perpendicular to AB. Show that AD : DB = $AC^2 . BC^2$.

22. In Fig. 468 X′OX, Y′OY are two straight lines at right angles. OA is equal to one unit of length; OP_1 is equal to x units; P_1P_2 is perpendicular to AP_1; P_2P_3 is perpendicular to P_1P_2, and so on. AQ_1 is perp to AP_1; Q_1Q_2 is perpendicular to AQ_1, and so on.

Show that $OP_2 = x^2$, $OP_3 = x^3$, $OP_4 = x^4$, and so on; and that $OQ_1 = \frac{1}{x}$, $OQ_2 = \frac{1}{x^2}$, $OQ_3 = \frac{1}{x^3}$, and so on. (*Army.*)

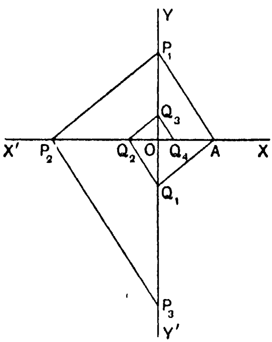


FIG. 468

23. The straight lines ABC and DE are parallel, and AB=BC. Draw BF parallel to CE and BG parallel to AD, to cut DE in F and G respectively; join AF to cut BD in H and CG to cut BE in K. Prove that HK is parallel to DE.

(Note that DF=GE.)

24. ABCD is a parallelogram, P and Q are the middle points of DC, CB respectively. AP, AQ meet DB in X and Y. Prove that BD is trisected at X and Y.

(If the diagonals meet at O, note that OP is || to AD.)

25. The tangents to a circle at P and Q meet in T, and C is the centre of the circle. TC meets the circle at A and B, and meets PQ at X.

Show that CX . CT = CA^2, and that AX : XB = TA : TB.

26. BX, CY, medians of a triangle ABC, meet at O. Prove that they trisect one another at O.

27. The rough sketch ABCDE of Fig. 469 is a section taken lengthwise through the middle of a swimming bath. Find the area of the section in square feet from the formula

$$\text{Area}=[p(x+y)+q(y+z)]\div 2,$$

where $p=15$ ft., $q=85$ ft.,

$x=3\frac{3}{4}$ ft., $y=6$ ft., $z=3\frac{1}{4}$ ft.

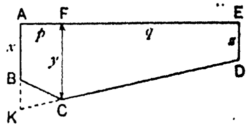


FIG. 469.

If the floor DC had been continued with the same slope down to K, what would have been the depth AK? (*Army*)

28. Draw a circle to touch two converging lines AX, AY and to pass through a given point P.

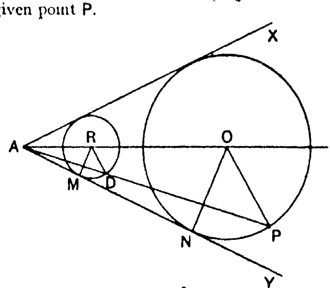


FIG. 470.

(Taking *any* point R on the bisector of the angle XOY, draw a circle with centre R to touch AY, at M. Join AP meeting this circle at D. Through P draw PO parallel to DR. Draw ON perp. to AY. Prove ON=OP.)

29. ABC is any triangle. Show how to inscribe a square PQRS in the triangle so that P lies on AB, Q on AC, and the side RS on BC.

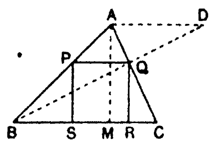


FIG 471

(Draw AM perp. to BC; AD perp. to AM, so that AD=AM. Join BD, cutting AC at Q. Draw QP ∥ to CB. Prove QR=PQ.)

30. ABCD is a cyclic quadrilateral. Prove that

$$AC \,.\, BD = AD \,.\, BC + AB \,.\, CD. \quad \text{(Ptolemy's Theorem.)}$$

(If E is a point on BD, such that angles BAE, CAD are equal, prove that the △^{s}BAE, ADC are similar, and also the △^{s}ADE, BAC. See Qn. 17.)

31. If ABC is a triangle, AD perpendicular to CB, and AE the diameter of the circumscribing circle, prove that $AB \,.\, AC = AD \,.\, AE$.

(Prove that the triangles ADB and ACE are similar.)

32. If the *internal* bisector AD of the angle A of the triangle ABC meets BC in D, prove that $AB \,.\, AC = BD \,.\, DC + AD^2$.

(If AD meets the circumcircle in E, prove that the △^{s}ABD and AEC are similar.)

33. If the *external* bisector AE of the angle A of the triangle ABC meets BC produced in E, prove that $AB \,.\, AC = BE \,.\, EC - AE^2$.

(If EA produced meets the circumcircle of ABC in F, prove that the △^{s}AFB, ACE are similar.)

34. Prove that, if ABC is an equilateral triangle and P any point on the arc BC of the circumscribed circle, then $PA = PB + PC$.

35. If three points X, Y, Z, lying on the sides of a triangle ABC, or the sides produced, are collinear, prove that

$$AZ \,.\, BX \,.\, CY = BZ \,.\, CX \,.\, AY. \quad \text{(Menelaus' Theorem.)}$$

(Draw BH || AC meeting XYZ in H.)

THEOREM 49.

If two triangles have their sides proportional, they are equiangular, those angles being equal which are opposite to corresponding sides, *i.e.* the triangles are similar.

Let ABC, PQR be two triangles, having

$$\frac{AB}{AC}=\frac{PQ}{PR},$$

$$\frac{AB}{BC}=\frac{PQ}{QR},$$

$$\frac{AC}{BC}=\frac{PR}{QR}.$$

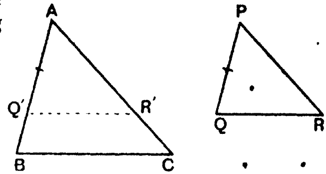


FIG. 472

We have to prove that the triangles are similar.

Construction. Along AB, produced if necessary, take AQ′ equal to PQ.

Draw Q′R′ ∥ BC, cutting AC at R′.

Proof. The △ˢ AQ′R′, ABC are equiangular;

$$\therefore \frac{Q'R'}{AQ'}=\frac{BC}{AB}.$$

But $$\frac{QR}{PQ}=\frac{BC}{AB};$$ (Hypoth.)

$$\therefore \frac{Q'R'}{AQ'}=\frac{QR}{PQ};$$

and since AQ′ = PQ, ∴ Q′R′ = QR.

Similarly, $$\frac{Q'R'}{R'A}=\frac{BC}{CA}=\frac{QR}{RP};$$

and since Q′R′ = QR, ∴ R′A = RP;

∴ △AQ′R′ ≡ △PQR. (3 sides)

But △ˢ AQ′R′, ABC are equiangular,

∴ △ˢ PQR, ABC are equiangular and similar.

THEOREM 50.

If two triangles have one angle of the one equal to an angle of the other, and also the sides about the equal angles proportional, the triangles are similar.

Let ABC, PQR be two triangles, having $\frac{AB}{AC} = \frac{PQ}{PR}$ and $\hat{A} = \hat{P}$.

We have to prove the triangles are similar.

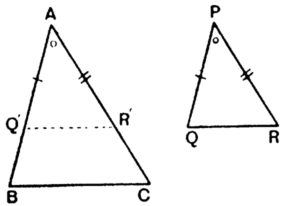


FIG. 473.

Construction. Along AB, AC, produced if necessary, take

$AQ' = PQ$ and $AR' = PR$. Join $Q'R'$.

Proof. Since $\triangle AQ'R' \equiv \triangle PQR$; (2 sides and inc. ang.)

$$\therefore \frac{AQ'}{AR'} = \frac{PQ}{PR} = \frac{AB}{AC}; \qquad \text{(Hypoth.)}$$

$\therefore$ $Q'R'$ is parallel to BC; (sim. division)

$\therefore$ $A\hat{Q}'R' = A\hat{B}C$, $A\hat{R}'Q' = A\hat{C}B$; (corr. angles)

i.e. $P\hat{Q}R = A\hat{B}C$, $P\hat{R}Q = A\hat{C}B$.

Thus the $\triangle^s$ ABC, PQR are equiangular;

$\therefore$ they are similar.

Ex. If the ratio of two sides of one triangle equals the ratio of two sides of another triangle, and if the angles *opposite* to one pair of these sides are also equal, prove that the angles opposite the other pair of sides must be either equal or supplementary.

THEOREM 51.

If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to one another.

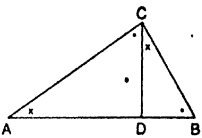


FIG. 471.

Let ABC be a right-angled triangle, with a right angle at C.

If CD is perp. to AB, it is required to prove that the $\triangle$'ACD, CDB, ABC are similar.

Proof. (i) In the $\triangle$'ACD, ABC,

$$\hat{ADC} = 90^\circ = \hat{ACB},$$

$$\hat{CAB} \text{ is common ;}$$

it follows that the third angles ACD, CBA are equal, and the triangles are equiangular and similar.

(ii) In the $\triangle$'CDB, ABC,

$$\hat{CDB} = 90^\circ = \hat{ACB},$$

$$\hat{CBA} \text{ is common ;}$$

$\therefore$ the third angles BCD, CAB are equal, and the triangles equiangular and similar.

(iii) Since the $\triangle$'ACD and CDB are equiangular to the $\triangle$ABC, they are equiangular to one another, and consequently similar.

COR. 1. *The square on the perp.* CD *is equal to the rect.* AD.DB *contained by the segments of the hypotenuse.*

For since the $\triangle$'ACD, CDB are similar,

$$\therefore CD : AD = DB : CD,$$

$$\therefore CD^2 = AD \,.\, DB.$$

COR. 2. *The square on either of the sides containing the right angle is equal to the rectangle contained by the*

hypotenuse and the segment of the hypotenuse adjacent to that side.

Since the $\triangle^s$ ACD and ABC are similar,

$$\therefore\ AC : AD = AB : AC, \quad i.e.\ AC^2 = AD \,.\, AB.$$

Similarly, it may be shown that

$$BC^2 = BD \,.\, BA.$$

It follows that $\quad AC^2 + BC^2 = AD \,.\, AB + BD \,.\, BA$

$$= AB\,(AD + BD)$$

$$= AB^2,$$

the well-known theorem of Pythagoras.

CONSTRUCTION 18.

To find a mean proportional between two given straight lines.

Let AB, BC be any two straight lines It is required to find a mean proportional between them.

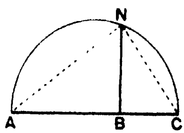


FIG. 475.

Construction. Place the lines in one and the same straight line.

On AC describe a semicircle.

Erect BN a perpendicular to AC.

Join AN and CN.

Proof. $\hat{ANC}$ being the angle in a semicircle is a right angle, and NB is perp. to AC,

$$\therefore \text{ the } \triangle^s\ BAN,\ BNC,\ NAC \text{ are similar.}$$

Because the $\triangle^s$ BAN, BNC are similar,

$$\therefore\ BA : BN = BN : BC, \quad i.e.\ BN^2 = BA \,.\, BC,$$

or BN is a mean proportional to AB and BC.

As we have already seen in the corollaries to the previous theorem, if the two lines are AB and AC, then,

$$AN^2 = AB \,.\, AC.$$

and AN is a mean proportional to AB and AC.

Similarly CN is a mean proportional to CB and CA.

This construction is that employed for describing a **square equal in area to a given rectangle** (page 221).

EXERCISE 47.

1. Draw the two triangles $\hat{A}=40°$, $b=3$, $c=4$, and $\hat{A}'=40°$, $b'=6$, $c'=8$. Measure the third side in each case; verify that one is double the other, and say why.

2. Draw two triangles with sides 2, 3, 4 and 4, 6, 8. Measure the greatest angle in each case, and explain your result.

3. By drawing a line AB = 7·5 cms, and cutting off AC = 2 cms., find by construction the square root of 15.
(Construct the mean proportional to 2 and 7·5.)

4. Find, by construction, the mean proportional to 4 and 5.

5. Describe a rhombus ABCD, each side of which is 2·64 inches in length, and two opposite angles BAD, BCD each 35°. Join the diagonals AC, BD intersecting in O. Find the length of a mean proportional between AO and BD.

6. The diagonals of a quadrilateral ABCD meet at H, and

$$AH \cdot BH = DH \cdot CH,$$

prove that the angles BAC, BDC are equal, and that the quadrilateral is cyclic.

7. In a triangle ABC, AD is drawn perpendicular to BC, and AD is a mean proportional between BD, DC, *i.e.* BD : DA = DA : DC; prove that A is a right angle.

8. ABCD is a quadrilateral; H is a point in BD such that AB : BD = CH : HB and AD : BD = CB : HB; prove that AB, HC and AD, BC are pairs of parallels.

9. P, Q, R are three collinear points, QH, RK are parallels and PQ : QH = PR : RK; prove that P, H, K are collinear.
(Prove that △s PQH, PRK are similar and ∴ equiangular.)

10. O is any point inside a triangle ABC, in OA, OB, OC points D, E, F are taken such that $\frac{OD}{OA}=\frac{OE}{OB}=\frac{OF}{OC}$; prove that the triangles ABC, DEF are similar.

11. ABC, PQR are similar triangles; D is a point in BC, and S is a point in QR, such that BD : DC = QS : SR. Prove that the angles ADB, PSQ are equal.

12. In the triangle PQR, PQ = PR, and the bisector of the angle R meets PQ at S; ABC is a triangle in which AB = AC = PS, and BC = QS; prove that the triangle ABC is similar to the triangle PQR.

13. B is the middle point of a straight line ABC, and E the middle point of a parallel line DEF; prove that AD, BE, CF will meet in a point or be parallel.

14. The base BC of an isosceles triangle ABC is produced both ways to D, E. Show that if $AB^2 = DB \cdot CE$, the triangles EAC, ADB are equiangular.

15. P is any point on a given circle, with centre C. O is any fixed point outside the circle. OC is produced to D, a fixed point, and OP is produced to Q, so that $OQ : OP = OD : OC$. Show that as P moves on the given circle, Q moves on another circle, whose centre is D.

16. A and B are the centres of two circles, and C is the point which divides AB in the ratio of the radii. A straight line drawn

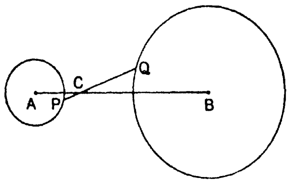


Fig. 476.

through C meets the two circles at P and Q as in the figure. Prove that PQ is divided in the ratio of the radii, and that the tangents to the circles at P and Q are parallel.

17. ABCD is a quadrilateral in which $\hat{A} = C\hat{B}D$; P is any point in AB, and $CB : BD = PA : AD$; prove that BAD is similar to CPD.

NOTES ON SIMILAR POLYGONS.

1. *Given any rectilineal figure and a finite straight line parallel to one of its sides, it is always possible to describe on the given line another rectilineal figure similar to the first, such that the joins of corresponding vertices are concurrent, and the corresponding sides are parallel.*

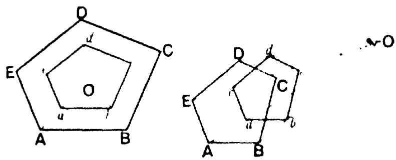


FIG. 477

Let ABCDE be any rectilineal figure, ab any finite straight line parallel to AB.

Join Aa, Bb, meeting at O.

Join OC, OD, OE.

Draw bc, cd, de parallel to BC, CD, DE, and join e to a.

We have to prove $abcde$ is similar to ABCDE.

Since
$$\frac{AB}{ab}=\frac{OA}{Oa}$$
$$=\frac{OB}{Ob}=\frac{OC}{Oc}=\frac{OD}{Od}=\frac{OE}{Oe};$$
$$\therefore \frac{OA}{Oa}=\frac{OE}{Oe}; \text{ i.e. } ea \text{ is } \parallel \text{EA}.$$

$\therefore$ ABCDE and $abcde$ have all their sides respectively parallel and are consequently equiangular.

Also any ratio such as $\dfrac{CD}{cd}=\dfrac{OC}{Oc}=\dfrac{CB}{cb}$.

Thus the corresponding sides have the same ratio.

Thus $abcde$ is similar to ABCDE, has all its sides parallel to the corresponding sides of ABCDE, and is such that the joins of corresponding vertices are concurrent.

If ab is placed along AB in the position Ab, and the point O taken at A, then, by drawing bc parallel to BC, cd parallel to CD, and de parallel to DE, it follows from the above that A$bcde$ is similar to ABCDE.

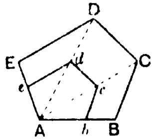


FIG. 478.

2. *Any two similar rectilineal figures can be placed in an infinite number of positions, with their corresponding sides parallel; and when in any such position the joins of their corresponding vertices must be concurrent.*

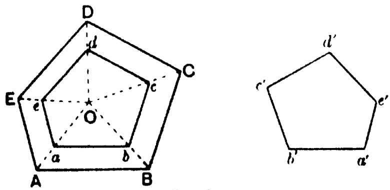


FIG. 479.

Let ABCDE and $a'b'c'd'e'$ be any two similar rectilineal figures.

Take a line $ab = a'b'$, parallel to AB and near it. Join Aa, Bb, meeting at O. Join OC, OD, OE, and make bc, cd, de, ea || BC, CD, DE, EA as in Note 1.

Then $abcde$ is similar to ABCDE,

and $\therefore$ $abcde$ is similar to $a'b'c'd'e'$.

Also $ab = a'b'$, $\therefore$ $abcde = a'b'c'd'e'$.

Hence $a'b'c'd'e'$ may be turned over and placed in the position $abcde$.

COR. Since any pair of triangles such as OBC and Obc are similar, it follows that if any rectilineal figure is divided into triangles by lines joining a point to its vertices, any similar figure can be divided into corresponding similar triangles.

EXERCISE 48.

1. Describe a triangle ABC with $a=5$, $b=7$, $c=8$ cms., and inscribe an equilateral triangle in it, with one side parallel to the longest side. Measure the sides.

(Draw any line DE ∥ to AB, meeting CA, CB in D, E respectively. On DE describe an equilateral △DEF. Produce CF to meet AB in P. Draw PQ, PR ∥ to FD, FE. Join QR. PQR is the required triangle.)

2. Inscribe in a given triangle another triangle having its sides parallel to three given directions.

3. Show how to draw a quadrilateral similar to a given quadrilateral, but having each side half the original length and having one side passing through a given point.

4. Describe a triangle $a=5$, $b=7$, $c=9$. Draw near C a line parallel to AB cutting AC, BC at P, Q. On PQ describe PQRS a square; join CR, CS, meeting AB in M, N. On MN draw a square having its other angular points on AC, BC. Measure its side.

5. In a pentagon, sides 4 cms long, inscribe a square having two sides parallel to a given side of the pentagon and one angular point on each other side. Measure its sides.

(Use a method similar to that of Qn. 4.)

6. Draw a sector of a circle with radius 5 cms. and angle 50°. In it inscribe a square with two angular points on the radii and the other two on the arc. Measure its side.

7. Describe a semicircle with radius 2·2 inches. Inscribe a square within the semicircle, one side lying on the diameter, and measure its side.

8. Construct a triangle the sides of which are in the proportion of 5, 6, 7, and the radius of its circumscribing circle is 2 inches in length. Measure the sides.

(Draw any triangle with its sides in the given ratios, and draw one of the radii of the circumscribing circle. Magnify the figure so that the radius becomes 2 inches.)

9. Construct a triangle the angles of which are in the proportion of the numbers 1, 2, 3, and the radius of its inscribed circle one inch in length. Measure the sides.

10. Draw a triangle ABC having $\angle B=27°$, $\angle C=44°$, AB=4·7 cms. Using only geometrical means, construct a triangle similar to the triangle ABC, and having the sum of its sides equal to 12 cms. Measure the sides.

AREAS OF SIMILAR FIGURES.

We have already seen that the areas of *any* two triangles with equal bases are proportional to the altitudes, and also that if the altitudes are equal, then the areas are proportional to the bases. We now come to the case of *similar* triangles.

THEOREM 52.

The areas of similar triangles are proportional to the squares on corresponding sides.

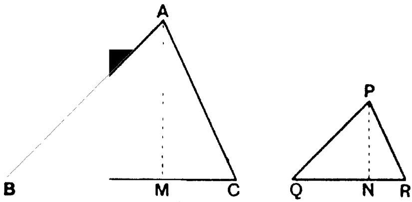


FIG. 480

Let ABC, PQR be two similar triangles.

We have to prove $\dfrac{\triangle ABC}{\triangle PQR} = \dfrac{BC^2}{QR^2}$ or $\dfrac{AB^2}{PQ^2}$ or $\dfrac{AC^2}{PR^2}$.

Construction. Draw AM, PN $\perp^{\text{rs}}$ to BC, QR respectively.

Proof. The triangles AMC and PNR are equiangular;

$\therefore$ they are similar.

Hence $$\frac{AM}{PN} = \frac{AC}{PR} = \frac{BC}{QR};$$

$$\therefore \frac{\triangle ABC}{\triangle PQR} = \frac{\frac{1}{2}AM \,.\, BC}{\frac{1}{2}PN \,.\, QR} = \frac{BC \,.\, BC}{QR \,.\, QR} = \frac{BC^2}{QR^2}.$$

COR. In the case of *similar polygons*, since they may be divided up into an equal number of similar triangles such as OAB, O*ab*, it follows that the *areas of similar polygons are proportional to the squares on corresponding sides.*

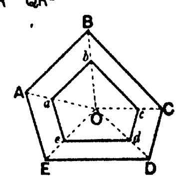


FIG. 481.

FURTHER NOTES ON AREAS.

1. *If two triangles have one angle of one equal to one angle of another, then their areas are proportional to the rectangles contained by the sides about these equal angles.*

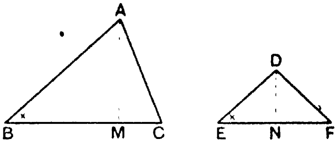


FIG. 482.

Let ABC, DEF be the two triangles, having $\hat{ABC} = \hat{DEF}$.

Draw AM, DN perp. to BC, EF respectively.

Since the triangles ABM and DEN are equiangular, it follows that they are similar;

$$\therefore \frac{AM}{DN} = \frac{AB}{DE};$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}AM \cdot BC}{\frac{1}{2}DN \cdot EF} = \frac{AB \cdot BC}{DE \cdot EF}.$$

A similar result holds for parallelograms having one angle of one equal to one angle of another.

2. *If similar rectilineal figures are described on the sides of a right-angled triangle, the area of the figure on the hypotenuse must be equal to the sum of the areas of the other two figures.*

Let ABC be a triangle right-angled at A, and X, Y, Z similar rectilineal figures described on the sides.

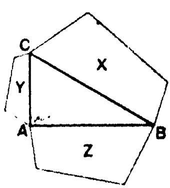


FIG. 483.

By Cor. (page 252),

$$\frac{Y}{X} = \frac{CA^2}{BC^2} \text{ and } \frac{Z}{X} = \frac{AB^2}{BC^2};$$

$$\therefore \frac{Y+Z}{X} = \frac{CA^2 + AB^2}{BC^2}.$$

But $CA^2 + AB^2 = BC^2$;

$$\therefore Y + Z = X.$$

3. *Describe a figure similar to a given rectilineal figure, and* n *times its area.*

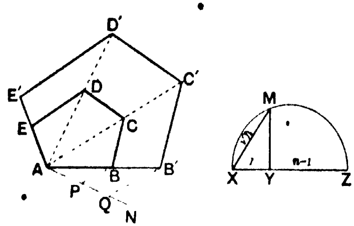


FIG 481.

Let ABCDE be the given figure.

Construction. Draw a line XY one unit long, and produce to Z so that XZ $= n$ units.

On XZ describe a semicircle, and draw YM perp. to XZ meeting the semicircle in M. Join XM.

From A draw AN at any angle, and cut off AP $=$ XY and AQ $=$ XM.

Join PB, and draw QB′ parallel to PB meeting AB produced in B′.

Draw B′C′ ‖ BC, meeting AC produced in C′,

C′D′ ‖ CD, ,, AD ,, D′,

D′E′ ‖ DE, ,, AE ,, E′;

then fig. AB′C′D′E′ $=$ fig. ABCDE $\times n$.

Proof. $XM^2 = XY \cdot XZ$ (angle XMZ $= 90°$)

$$= 1 \cdot n = n.$$

Since fig. AB′C′D′E′ is similar to ABCDE,

$$\therefore \frac{\text{fig. } AB'C'D'E'}{\text{fig. } ABCDE} = \frac{AB'^2}{AB^2} = \frac{AQ^2}{AP^2} = \frac{XM^2}{1}$$

$$= n.$$

Theorems 43 and 44 may be proved by the method of similar triangles.

1. *If two chords of a circle intersect at a point within the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.*

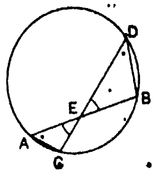


FIG. 485.

Let AB, CD be two chords of a circle intersecting at E; we have to prove that

$$AE \cdot EB = CE \cdot ED.$$

Construction. Join AC, BD.

Proof. $C\hat{A}B = C\hat{D}B$ (angles in same segment)

$C\hat{E}A = B\hat{E}D$, (vertically opposite)

$\therefore$ $\triangle$'AEC, BED are equiangular and similar;

$\therefore$ $AE : EC = ED : EB$, $\therefore$ $AE \cdot EB = CE \cdot ED$.

2. *If from a point without a circle, a secant and a tangent to the circle be drawn, the rectangle contained by the whole secant and the segment of it without the circle is equal to the square on the tangent.*

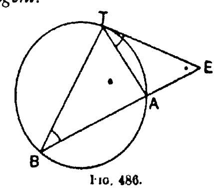


FIG. 486.

Let ET be a tangent to the circle and EAB a secant; we have to prove that

$$EA \cdot EB = ET^2.$$

Construction. Join AT, TB.

Proof. $E\hat{T}A = A\hat{B}T$, (angle in alt. segment)

$T\hat{E}B$ is common,

$\therefore$ $\triangle$'TEA, BET are similar;

$\therefore$ $EA : ET = ET : EB$; $\therefore$ $EA \cdot EB = ET^2$.

EXERCISE 49.

1. A map is drawn on a scale of 2 inches to the mile. If the area of a county on the map is 2·5 sq. feet, what is the actual area of the county ?

2. The area of the plan of a field is 20 sq. inches, and it is drawn to a scale of 2 inches to a mile. What is the real area of the field ?

3. If the corresponding sides of two similar triangles are 2·3 cms. and 3·2 cms. in length, what is the ratio of the areas of the triangles ?

4. The area of one of two similar triangles is 9 times the area of the other, and a side of the latter is 2·7 cms in length. Calculate the length of the corresponding side of the first triangle.

5. Construct a triangle ABC such that BC = 8 cms., CA = 5 cms., AB = 7 cms.

Find a point D in BC, such that if DE be drawn parallel to CA to meet AB in E, the area of the triangle BDE shall be half that of the triangle ABC. Measure BD

(Find D such that DB : BC = 1 : $\sqrt{2}$)

6. Draw a triangle ABC such that AB = 2 cms, AC = 3 cms., BC = 4 cms. On AB and AC produced, take points D and E, such that $\triangle$ADE = 9$\triangle$ABC

Measure the length of the perpendicular from A to DE.

7. Two triangles ABC and DEF are such that $\hat{B} = \hat{E} = 32°$, AB = 3 cms, BC = 5 cms, ED = 4 cms, EF = 7 cms. What is the ratio of the areas of the triangles ? Draw the triangles, and measure the lengths of the perpendiculars from A and D on to BC and EF respectively. What is the ratio of these perpendiculars ?

8. Construct a triangle with sides 3·2, 2·3 and 4·1 cms. long; then construct a similar triangle three times its area, and measure the length of the sides.

9. If ABC is a triangle and DE be drawn parallel to BC meeting AB and AC in D and E respectively so that AD = $\frac{1}{4}$AB, find the ratio $\triangle$ADE : $\triangle$ABC.

10. If ABC is a triangle right-angled at A, and a line be drawn from A perpendicular to BC, show that the triangle ABC will be divided into two triangles which are to one another as $AB^2 : AC^2$.

11. AB and XY are two chords of a circle intersecting in a point O external to the circle. Prove that

$$\triangle OBY : \triangle OAX = BY^2 : AX^2.$$

12. AD and BE are the perpendiculars on the sides BC, AC of the triangle ABC; prove that the triangles CDE, ABC are in the ratio of CD^2 to CA^2.

13. Prove that the areas of two similar cyclic polygons are in the ratio of the squares of the corresponding diameters of the circum-circles.

14. TP, TQ are tangents to a circle at any points P, Q, and O is the centre, and the straight lines OP, OQ, PQ are drawn. Prove that the area of the triangle OPQ : the area of the triangle TPQ = square on OP : square on PT.

15. Two circles with their centres at X and Y touch externally at A; any straight line through A meets the circles again in B and C. Prove that the triangles ABY and ACX are equal in area.

16. Draw a circle with radius 4 cms, and then draw two more concentric circles which divide the area of the original circle into three portions of equal area. Measure the two radii.

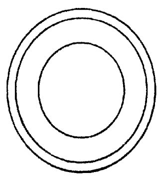


FIG 487.

17. Describe a rectilineal figure similar to one figure X and equal in area to another figure Y.

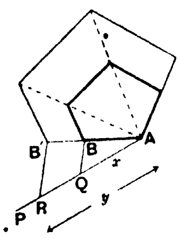


FIG. 488.

(Reduce the figures to squares whose sides are x and y respectively; if AB is one of the sides of the first figure, draw AP at any angle and mark off AQ$=x$, AR$=y$; join QB, and draw RB′ parallel to QB; then B′ is one of the corners of the required figure.

$$\frac{\text{New figure}}{X}=\frac{y^2}{x^2}=\frac{Y}{X}; \quad \therefore \text{ new figure}=Y.)$$

18. Construct a regular hexagon equal in area to an equilateral triangle of sides 1·5 inches. Measure its sides.

19. Construct a rectangle with its sides in the ratio of 2 to 1 and whose area is 5·2 sq. inches. Measure the sides.

20. Construct a triangle with sides 3, 4, 5·2 cms., and another with sides 3·5, 4·5, 5·8 cms. Make a triangle similar to the first and equal to the second, and measure its sides.

21. In any $\triangle ABC$, if X, Y, Z are points on the sides, or sides produced, such that AX, BY, CZ are concurrent, prove that

$BX \cdot CY \cdot AZ = CX \cdot AY \cdot BZ$. (Ceva's Theorem.)

ANSWERS.

PART I.

Exercise 3. (Page 11.)

1. 1 rt. ∠. 2. $\frac{1}{2}$ rt ∠ 3. $\frac{3}{2}$ rt ∠ 4. 2 rt. ∠s.
5. $\frac{4}{3}$ rt ∠. 6. $\frac{4}{3}$ rt ∠ 7. $\frac{1}{2}$ rt ∠ 8. $\frac{3}{2}$ rt. ∠.

Exercise 4. (Page 14.)

1. AB = 1·26, BC = 2·56, CD = 1·23, AD = 5·05.
2. AD = 5·75 ; AC = 4·5 ; AB = 2·28.
3. 0·5 ; 1 ; 0·5 ; 2. 4. 2·3, 1·8, 0·9.
6. 8·5 cms. ; 3·3 ins. 7. 8·9 cms., 3·5 ins.
8. BD = DC = 2·2 cms. ; CE = EA = 1·5 cms., AF = FB = 2·1 cms. ;
GD = $\frac{1}{2}$GA = 1·0 cms. ; GE = $\frac{1}{2}$BG = 1·3 cms., FG = $\frac{1}{2}$GC = 1·1 cms.
9. 7 miles. 10. 2 inches. 12. 1·53 ins.
13. 3·0 cms. 14. 3·15 cms.
15. 4·0, 5·3, 4·7 cms. 16. 1·26, 2·08, 1·60 ins.

Exercise 5. (Page 18.)

1. AOB = $16\frac{1}{2}°$; BOC = 36°. 2. BOC = 36°, COD = 25°.
3. COD = 25° ; DOE = 70.° 4. DOE = 70°, EOF = $31\frac{1}{2}°$.
5. AOD = $78\frac{1}{2}°$; DOF = $101\frac{1}{2}$ 6. AOC = $53\frac{1}{2}°$; COF = $126\frac{1}{2}°$.
7. AOE = $148\frac{1}{2}°$; EOF = $31\frac{1}{2}°$. 8. AOB = $16\frac{1}{2}°$; BOF = $163\frac{1}{2}°$.
9. 63° ; 63° ; 2·7 cms. 10. 137° ; 27° ; 2·4 cms.
11. $119\frac{1}{2}°$; $38\frac{1}{2}°$; 3·0 cms. 12. 86° ; 56° ; 3·7 cms.
13. 33° ; 33°. 14. (i) 90°, (ii) 120° ; (iii) 150° ; (iv) 210°.
15. 126° ; 54° ; 126°. 16. 47° ; 56° ; 77°.

17. 42° ; 57° ; 81°. **18.** AD = 5·6 miles, BAD = 52°.

19. 2·6 ins. **20.** AQ = 1·45 ins., QAB = 21°.

22. OA = OB = OC = 3·3 cms. ; OX = 0·4 cm., OY = 1·9 cms. ; OZ = 2·5 cms.

23. 8·5, 8·0, 6·8, 5·4, 4·3, 3·6, 3·5 cms.

Exercise 6. (Page 22.)

1. 23°. **2.** 90°. **3.** 3·6 cms.

4. 2·1 cms ; 1·9 cms. **6.** 1·04 ins., 0·96 ins. **7.** 2·27 ins.

8. 1·9 ins. ; 62°. **10.** OB = OC = 3·29 ins. **13.** 75°, 118°.

15. 3·5 cms., 1·3 cms. **16.** 3·9 cms. **17.** 2·3 ins. ; 45°.

18. 90 ft. **19.** AC = 150 ft., AB = 155 ft. **20.** 156 ft.

Exercise 7. (Page 25.)

1. 38°, 190°, 226°, 113°, 324°, 272°.

2. 135°, 315°, 225°, $157\frac{1}{2}^{\circ}$, $112\frac{1}{2}^{\circ}$, $337\frac{1}{2}^{\circ}$, $247\frac{1}{2}^{\circ}$, $22\frac{1}{2}^{\circ}$, 130°, 220°, 30°, 312°.

3. (i) 73°, 27°, 141°, 71°, (ii) 347°, 87°, 207°, 307°.

4. 5 miles, 323°. **5.** 127°. **6.** 223°. **7.** $20\frac{1}{2}^{\circ}$.

8. 45°, 225°, 3470 yds. **9.** 1060 yds., $175\frac{1}{2}^{\circ}$.

10. 168°, 194°, 870, 670 yds. **11.** 65 yds. **12.** 770 yds.

Exercise 8. (Page 28.)

1. 5·06 miles. **2.** 17·8 miles, N. $35\frac{1}{2}^{\circ}$ E. **3.** 11·1 miles.

4. 532 yds. **5.** $28\frac{1}{2}^{\circ}$. **6.** 325 ft. **7.** 9·79 miles.

8. 10·4 ft. **9.** 154 ft. **10.** 24·5 ft. **11.** 8·13 miles.

12. 12·4 miles ; S. $30\frac{1}{2}^{\circ}$ W. **13.** 14·2 miles. **14.** 14·6 miles.

15. 171 ft. **16.** 536 ft. **17.** 245 ft. **18.** 87·2 yds.

19. 24·8 yds. **20.** 42·2 ft. **21.** 17·1 miles.

22. 648 ; 1340 yds. **23.** 291 ft. **24.** 105 ft.

25. 13·4 ft. **26.** 27·0 miles. **27.** 332 ; 372 yds.

28. 1480 yds. ; 352°. **29.** 16·2 yds. **30.** 73·7 ft.

Exercise 10. (Page 45.)

10. 63°, 42°, 42°, 63°.

19. Each 90° ; 5, 8·7 cms.

Exercise 11. (Page 48.)

1. 3·8... cms. 2. 1·58... ins. 3. 3·69 ins.

4. 3·1 ins. 5. Each 1·7 cms.

6. AR = RC = 2·3 cms. ; AS = SB = 1·9 cms.

7. 8·2 cms. 8. 7·2 cms.

Exercise 12. (Page 51.)

9. $(90-x)^\circ$. 10. 45° 12. 78°.

13. 60°, 100°. 15. 140°, 150°, 70°. 17. 33°.

19. 114°, 124°, 122°. 20. 56°, 42°, 82°. 21. 70°, 43°.

22. 96°, 40°, 8°, 82°. 23. 9·2, 10·8 cms 24. 145°, 105°.

Exercise 13. (Page 55.)

5. 9. 6. 12 7. 10. 8. 60°, 80°, 100°, 120°.

9. 95°. 10. 7. 11. 120°. 12. 36°, 108°, 36°, 36°.

Exercise 14. (Page 58.)

1. 2·3 cms. ; 94°, 45°. 2. 7 cms. ; 30°, 88°.

3. 3·3, 3·7 cms. 5. 70°, 70°, 90°, 90°.

26. PQ = 1 inch

Exercise 16. (Page 66.)

1. 70°. 2. 63°. 3. 19° 4. 45°.

5. 65°, 65° or 50°, 80°. 6. 74°, 106°. 7. 45°

8. 70°, 40°. 9. 3·1 cms. 10. 39°.

11. 17°, 3·1 cms. 24. 6·1 ins. 26. 72° ; 54°.

27. (i) 60° ; 60° ; (ii) $51\frac{3}{7}^\circ$; $64\frac{2}{7}^\circ$; (iii) 45° ; $67\frac{1}{2}^\circ$.

29. 15°, 15°, 20°, 20°, 55°, 55°. 30. $M\hat{K}A = 3M\hat{C}A$.

Exercise 17. (Page 69.)

1. 22°, 38°, 120°. 2. 28°, 46°, 106°.

5. 60° ; 4·3 cms. 6. 1·3, 1·9, 2·1 cms.

Exercise 18. (Page 72.)

1. 12 cms. 2. 7·4 cms.; 90°. 3. 5·5 cms.

Exercise 19. (Page 77.)

1. (i) 4·6, 7·5 cms.; (ii) 6·1 cms.; (iii) impossible
2. (i) 0·7, 5·4 cms.; (ii) 3·1 cms.; (iii) impossible.
3. (i) 1·4, 10·9 cms.; (ii) 6·1 cms.; (iii) impossible.
4. 2·4 cms. 5. 2·7, 8·9 cms.; c = 4·9 cms., A = 90°.
6. B, 3·3 cms.; 68°; 2·3, 6·3 cms.
7. 9·6, 4·8, 2·5, 2 cms. 8. Each = $\frac{1}{2}$.

Exercise 20. (Page 84.)

1. 3·8. 2. 2·7, 4·3, ratios equal 3. 2·9, 4, 4.
4. 3·6. 5. 3·2, 4·7, 90°. 6. 1·9. 7. 27$\frac{1}{2}$, 27$\frac{1}{2}$.
8. 2, 2, 2. 9. 29° 10. 3, 1·5 ins. 11. 1·97 ins.
12. 1·9 ins. 13. 25°, 1·4 ins. 14. 4·8 ins. 15. 7·8, 10·9 ins.
16. 4·4. 17. 86$\frac{1}{2}$°, 60° 18. 10·7. 19. 8·5.
20. 11·4. 21. 2·8, 1·8 ins.
22. (i) 1160, 860 yds.; (ii) 1320, 1240 yds.; (iii) 430, 510 yds.; (iv) 300, 330 yds.
23. 2·6 ins. 24. 13·6 ins., 5 ins.

Exercise 21. (Page 90.)

1. 30°, 94°. 2. 19°, 99$\frac{1}{2}$°. 3. 4·5, 6·8 cms.
4. (i) b, a, c; (ii) c, a, b; (iii) a, c, b; (iv) a, c, b. 19. 13 cms.

Exercise 22. (Page 99.)

1. AO = CO = 2·8 ins.; BO = DO = 1·2 ins.
2. AC = DB = 9·2 cms.; AD = 15·3 cms.; MC = MB = 3 cms.; MA = MD = 7·6 cms.
3. 1·7 ins., 2 ins. 4. 3·6 ins. 5. 2·0 ins.
6. 70°, 110°; 4·3 cms. 7. 2·1, 4·4 cms. 8. 4·8, 10·8 cms.
9. 5·9 cms. 13. Each = 1·9 ins. 25. 64°.

ANSWERS

Exercise 23. (Page 106.)

1. 34°, 65°.	**2.** 0·71 inch.	**3.** 4·6 cms.
4. 5·3, 5·0, 3·9 cms.	**5.** 2·9, 6·4, 5·7 cms.	**6.** 14 cms.

Exercise 25. (Page 115.)

4. 4·4 cms.	**5.** 7·1 cms.	**6.** 4·9.
7. 1·9 ; 2·1, 2·9	**8.** 1·6	**9.** 5·2 cms.
15. 5·0, 8·0 cms.		

Exercise 26. (Page 125.)

1. 17·1 sq. cms.	**2.** 16·4 , 4·6.	**3.** 4·4, 7·3 cms.
4. AC=1·7 or 4·3 ins , area=1·8 sq. ins.		
5. 19·4 ; 3·2.	**6.** 11·5 sq cms	**7.** 8·4 sq. ins.
8. 3·5, 10·6 cms.	**9.** 19·7 cms.	**10.** 8·9 sq. cms.
11. 5·7, 7·4 cms.	**12.** 22·2 sq. cms	**13.** 7·2 cms.
14. 6·3, 11·4 cms.	**15.** 8·8, 10·9 cms.	**16.** 22·5 sq. cms.
17. 15 sq. cms.	**18.** 21 sq cms	**19.** 20·25 sq. cms.
20. 40·04 sq cms.	**21.** 108 sq. cms.	**22.** 212 sq. cms.
23. 300 sq. ins	**24.** 158 sq. ins	
25. 1013 sq. yds ; 806 sq m ; 66,200 sq links		
26. 5, 8 cms.	**27.** 10,600 sq yds.	**28.** 62·4 sq. cms.
29. 6·4 sq. ins.	**30.** 2·5 cms.	**31.** 49·7 sq. cms.
32. 1·85 cms.	**33.** 20·7 sq. cms	**34.** 44·8 sq. cms.
35. 5·2 cms.	**36.** 3·2 cms.	**37.** 3·7 cms.
39. DE=3·9 cms. ; EC=4·2 cms.		**40.** 4·7, 11·4 cms.
41. 7·2 cms.	**42.** 5 cms.	
43. 6 cms.	**44.** 1·4, 1·8, 2·2 ins.	

Exercise 27. (Page 138.)

1. (i) 7·8 cms. ; (ii) 5 cms. , (iii) 7·1 ft.

2. (i) 6 cms. ; (ii) 7·2 cms. ; (iii) 8·3 cms.

3. 6·2 ins. **4.** 18 ft. $4\frac{1}{2}$ ins. **5.** 3·16, 9·54, 5·29, 6·08.

6. 7·81. **7.** 3·32. **8.** 3 ins. **9.** 85·9 ft.

10. (i) 21 ft. 1 in.; (ii) 17 ft.; (iii) 22 ft. 2 ins.

11. 8·71 miles. **12.** 4·24 miles. **13.** 24 ins.; 168 sq. ins.

14. 52·39 sq. ins. **15.** 15 ft. **16.** 54·1 ft.

17. 10 ft. **18.** 6·27 cms.; 33 sq. cms. **19.** 7·5 cms.

20. 3·3 ins.; 4·0 ins. **21.** 3·36 ins.

22. (i) 218 sq. ft.; (ii) 94·3 sq. ft. **23.** 3·39 ins.; 31·25 sq. ins.

24. 7 ft. **25.** 16·3 ft.; 13·9 ft. **26.** 4·18 ins

27. (i) 24·2 ft.; (ii) 23·4 ft.; (iii) 16·8 ft.

43. 8·7 cms.; 13·2 cms.

PART II.

Exercise 28. (Page 150.)

1. 4 cms. 2. 3·62 cms. 3. 6·20 cms.
4. 4·58, 4·33, 4 cms. 5. 11 ft , 140°. 6. 1·8 cms.
7. 2·7 cms 8. 6·8 cms. 9. 4·1 cms.
11. 5·4 cms ; 90°. 12. 2·1 ins. 13. 8·8 cms.
14. 2700 yds. , 64°. 15. (i) 1·13 cms. ; (ii) 6·56 cms.
16. 4·88 cms.

Exercise 29. (Page 159.)

1. $P\hat{A}Q = 38°$ or $102°$; $P\hat{S}Q = 76°$ or $156°$; PQ = 3·9 or 4·8 cms.
2. Each = 1·3. 3. 55°, 110°. 4. 40°, 100°.
6. 1·3 ins. 7. 4·3 cms. 8. 4·2 cms.
9. 2·3 ins. 10. 10·2 ins 11. 3·2, 7·4 cms.
12. $29\frac{1}{2}°$, $110\frac{1}{2}°$. 13. 2·2 ins.
14. 5, 8·6, 10·8 miles. 15. 320 yds. 16. 25 chains.

Exercise 30. (Page 167.)

1. 2·2. 2. 4·5. 3. $\hat{B} = 70°$, $\hat{D} = 78°$; cyclic.
4. $A = 70\frac{1}{2}°$, $B = 81\frac{1}{2}°$, $C = 109\frac{1}{2}°$, $D = 98\frac{1}{2}°$.
5. 33°. 10. 150°. 27. $59\frac{1}{2}°$.

Exercise 32. (Page 175.)

1. 44, 22, 29·5 cms.
2. 154 sq. ft. ; 227·07 sq. m. ; 260·26 sq. cms.
3. (i) 39·8 cms. ; (ii) 31·8 ins. ; (iii) 0·83 yd.
4. (i) 5·64 yds. ; (ii) 4·32 m. ; (iii) 3·91 ins.
5. 6111 yds. 6. $4\frac{23}{63}$ = 4·37 ins. 7. $3249\frac{6}{7}$ miles.
8. (i) $9\frac{23}{28}$ = 9·82 sq. ins. ; (ii) $14\frac{47}{252}$ = 14·19 sq. ins.

9. $2187\frac{3}{7}$ sq. ft. **10.** 114 sq. ft.

11. (i) $227\frac{1}{14}$ = 227·1 sq. cms. ; (ii) $107\frac{1}{14}$ = 107·1 sq. cms.

12. 18 sq. ins. **13.** 30828 sq. yds.

14. 63 ft. **15.** 0·055 cu. cm.

17. (i) $\frac{11}{14}$ = 0·79 sq. ft. ; $\frac{3}{14}$ = 0·21 sq. ft.
(ii) Coloured, 2·28 sq. ft. ; clear = 1·72 sq. ft.
(iii) 0·285.

Exercise 33. (Page 183.)

1. 3·2 cms. **2.** 2·1 cms. **3.** 3·3 cms. ; 40°.

4. 100° ; 108°. **5.** 4·8 cms. **6.** 6·6 cms.

7. 76°, 118°, 96°, 70°. **8.** 3·6 cms. **9.** 3 ins.

10. 1·7 ins. **11.** 6·0 cms. **12.** 2·8 cms.

13. 7·2 cms. **14.** 0·74, 3·15 ins. **15.** 2·1 cms.

16. 1·5 ins. **17.** 5·83 cms.

Exercise 34. (Page 188.)

1. 1·73 ins. **2.** 5·2 cms. **3.** 7·1 cms.

4. 4·5 cms. ; 17·6 sq. cms. **5.** 1·39 ins.

6. 3·03 ins. **7.** 2·0 cms. ; 19·1 sq. cms.

8. (i) 2·5 cms. ; (ii) 2·1 cms. **9.** (i) 2·2 cms. ; (ii) 1·6 cms.

10. (i) 1·62 ins. ; (ii) 1·78 ins. **11.** 0·92 inch.

12. 0·88 inch. **13.** 2·2 cms.

Exercise 35. (Page 192.)

3. 2, 3, 4 cms. **7.** 3·1 cms. **8.** 1·5 cms.

Exercise 36. (Page 194.)

4. (i) AB = 680 yds., BC = 1270 yds. ; (ii) 1670 yds.

5. 4·6, 3·9 cms. **10.** 6·9 ins. **11.** 1·2 ins. **12.** 2·05 ins.

13. 2·7 ins. **14.** 3·7 cms. **15.** 1·7 ins.

16. 1·5 cms. **17.** 2·6 ins. **18.** 2·2 ins.

19. 13·4 cms. **20.** 3·2 cms. **21.** 6·5 cms.

22. 3·7 cms. **23.** 1·9, 2·3, 2·9 cms. **24.** 2·2 cms.

Exercise 37. (Page 200.)

1. $B\hat{T}A = 22°$, $T\hat{B}A = 138°$, $B\hat{C}A = 20°$, $B\hat{A}C = 118°$.

2. $A\hat{T}B = 35°$, $B\hat{A}T = 35°$, $A\hat{B}T = 110°$, $B\hat{A}C = 75°$, $A\hat{B}C = 70°$.

3. $B\hat{E}D = 10°$, $E\hat{D}B = 140°$, $B\hat{E}F = 90°$, $F\hat{B}E = 50°$, $B\hat{F}E = 40°$, $F\hat{B}A = 90°$.

4. $\hat{D} = 69°$, $\hat{E} = 61°$, $\hat{F} = 50°$

Exercise 38. (Page 204.)

1. 6·6 cms	2. 4·8 cms	3. 7·7 cms.
4. 4·8 cms.	5. 4·5 cms	6. 7·75 cms, 5·29 cms

Exercise 39. (Page 208.)

5. $2\frac{1}{3}$, $3\frac{2}{3}$ ins.	6. 2, 5 ins
13. AC = CB	14. Bisect the line

Exercise 40 (Page 212)

1. (*a*) Acute, (*b*) obtuse, (*c*) right angled.

2. (*a*) Obtuse, (*b*) right, (*c*) obtuse, (*d*) obtuse-angled.

3. 7·9, 7·5, 6·9, 6·1 cms.	4. 3·8, 6·0, 6·6 cms
5. 9·6 cms	6. 4·5, 6·9, 5·9 cms
8. 23·3 cms.	25. 2 cms

Exercise 41. (Page 217.)

1. 36 sq. cms., 12·7 cms		2. $3\frac{1}{2}$.	3. 5·66 ins.
5. 3·46 cms.	6. 3·16 ins	7. 40·79 ft.	8. 8 cms
9. Equal.	11. 5 cms	12. 110 miles	13. 13·4 miles.

14. (i) 29·4 ; (ii) 32·8 ; (iii) 18 , (iv) 31·9 ; (v) 52·7 ; (vi) 73·1 miles.

15. 30·9 miles.

Exercise 42. (Page 223.)

1. 3·3 cms.	2. 3·1 cms.	4. 7·3, 2·2 cms.	5. 3·3 cms.
6. 2·1 ins.	7. 1·7, 4·7 ins.	8. 2·8 ins.	
9. 3·2 ins.	10. 2·0 ins.	11. CD=89·4 ft.	
12. 2·5 cms.	13. 3·6 cms.	14. 1·3 or 4·2 cms.	

SHORTER GEOMETRY

Exercise 43. (Page 228.)

1. (i) $\frac{4}{5}=0 \cdot 8$; (ii) $\frac{1}{3}=0 \cdot 33$; (iii) $\frac{2}{7}=0 \cdot 29$; (iv) $\frac{1}{3}=0 \cdot 33$.

2. (i) 50·32 ; (ii) 0·47 ; (iii) 0·06.

3. A, 1·21 ins., 3·1 cms., 31 mms. ; B, 1·88 ins., 4·75 cms., 47·5 mms. ; $\frac{A}{B}=0 \cdot 66$.

4. A, 0·26 sq. ins., 1·7 sq. cms. ; B, 0·4 sq. ins., 2·6 sq. cms. ; $\frac{A}{B}=0 \cdot 65$.

5. (i) $2\frac{6}{7}$; (ii) 5 ; (iii) 12.

6. (i) 40 lbs. ; (ii) 24 ozs. ; (iii) 5 cu. ft.

7. (i) $\frac{3}{6}=\frac{8}{16}$ or $\frac{3}{8}=\frac{6}{16}$, (ii) $\frac{a}{c}=\frac{d}{b}$ or $\frac{a}{d}=\frac{c}{b}$.

8. 2·81, 4·69 cms. 9. 3·9, 9·1 ins. 10. (i) $21\frac{1}{3}$; (ii) 15.

Exercise 44. (Page 232.)

1. 1·2, 2 ins. 2. 1·04, 3·66 ins. 3. 2·3, 3·9 cms.
4. 6·6, 10·9 cms. 5. 16·8, 10·1 cms. 7. 3·5 cms.
8. Internal, 1·3, 3·4 cms. ; ext. 3·1, 7·8 cms. ; 2·33 : 1.
9. 1·7, 2·5, 4·2 cms. 10. 0·53, 1·6, 2·67 ins.
11. 6·67. 12. 7·5. 13. 5·33.
14. 6·25. 15. 5·3 cms. 16. 2·5 cms.

Exercise 45. (Page 235.)

1. 2·9, 1·1, 6·7, 2·7 cms. ; 5 : 2. 3. 1·65, 2·1 cms., or 3·6, 4·6 cms.
4. 1·8, 2·7 ins. or 1·2, 1·8 ins. 9. 9·4 cms.

Exercise 46. (Page 238.)

1. 2·6, 4·6 cms. 2. 5·2, 6·5 cms. 3. 100 ft.
4. 10 ft. 5. 3·76 cms. 6. 1·16 cms.
7. $20\frac{6}{11}=20 \cdot 5$ yds. 8. 720 yds. 9. 9·5 cms.; 6·4 cms.
10. 3·50, 7·03, 7·85 ft. 27. $476\frac{7}{8}$ sq. ft. ; $6\frac{15}{34}=6 \cdot 44$ ft.

Exercise 47. (Page 247.)

1. 2·6, 5·2 cms. **2.** 104°. **3.** 3·9 cms.
4. 4·5. **5.** 2·0.

Exercise 48. (Page 251.)

1. 3·1 cms. **4.** 2·7. **5.** 4·2 cms. **6.** 2·4 cms.
7. 1·95 ins. **8.** 2·8, 3·4, 4·0 ins.
9. 2·7, 4·7, 5·4 ins. **10.** 2·6, 4·0, 5·4 cms.

Exercise 49. (Page 256.)

1. 90 sq. m. **2.** 5 sq. m.
3. 529 : 1024 or 1 : 1·94. **4.** 8·1 cms.
5. 5·7 cms. **6.** 4·4 cms.
7. 15 : 28 ; 1·6, 2·1 cms. ; 3 : 4. **8.** 5·5, 4·0, 7·1 cms.
9. 1 : 16. **16.** 2·3, 3·3 cms. **18.** 0·61 ins.
19. 1·61, 3·22 ins. **20.** 3·4, 4·5, 5·7 cms.

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